

CORRELATION BASED PASSIVE IMAGING WITH A WHITE NOISE SOURCE

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ABSTRACT. Passive imaging refers to problems where waves generated by unknown sources are recorded and used to image the medium through which they travel. The sources are typically modelled as a random variable and it is assumed that some statistical information is available. In this paper we study the stochastic wave equation $\partial_t^2 u - \Delta_g u = \chi W$, where W is a random variable with the white noise statistics on \mathbb{R}^{1+n} , $n \geq 3$, χ is a smooth function vanishing for negative times and outside a compact set in space, and Δ_g is the Laplace–Beltrami operator associated to a smooth non-trapping Riemannian metric tensor g on \mathbb{R}^n . The metric tensor g models the medium to be imaged, and we assume that it coincides with the Euclidean metric outside a compact set. We consider the empirical correlations on an open set $\mathcal{X} \subset \mathbb{R}^n$,

$$C_T(t_1, x_1, t_2, x_2) = \frac{1}{T} \int_0^T u(t_1 + s, x_1) u(t_2 + s, x_2) ds, \quad t_1, t_2 > 0, \quad x_1, x_2 \in \mathcal{X},$$

for $T > 0$. Supposing that χ is non-zero on \mathcal{X} and constant in time after $t > 1$, we show that in the limit $T \rightarrow \infty$, the data C_T becomes statistically stable, that is, independent of the realization of W . Our main result is that, with probability one, this limit determines the Riemannian manifold (\mathbb{R}^n, g) up to an isometry. To our knowledge, this is the first result showing that a medium can be determined in a passive imaging setting, without assuming a separation of scales.

1. INTRODUCTION

In passive imaging, waves generated by unknown noise sources are recorded and used to image the medium through which they travel. Passiveness refers to the observer having only little or no control over the source (think earthquakes in seismic imaging). However, some statistical information of the noise may be available and it can be useful to model the noise source as a random process: while the statistics of the source process is known, its realization remains unknown.

Passive imaging has had a fundamental impact to seismic and various other imaging modalities. We refer to the recent book by Garnier and Papanicolaou [18] for

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an extensive review of the field. The previous mathematical theory is, to a large extent, based on assuming some physical scaling regime. Such an approach has also produced a number of important and efficient numerical methods. However, our key finding in the present paper is that exact recovery is also possible without any scaling assumptions. Our hope is that results like this can help to build a framework for passive imaging where the imaging problems are reduced to deterministic inverse problems.

In this work we consider the wave equation

$$(1) \quad \begin{aligned} \partial_t^2 u(t, x) - \Delta_g u(t, x) &= \chi(t, x) W(t, x) \quad \text{in } \mathbb{R}_+^{1+n} = (0, \infty) \times \mathbb{R}^n, \\ u|_{t=0} &= \partial_t u|_{t=0} = 0, \end{aligned}$$

where $n \geq 3$ and Δ_g is the Laplace–Beltrami operator corresponding to a smooth time-independent Riemannian metric g on \mathbb{R}^n . In coordinates $(x_j)_{j=1}^n$ this operator has the following representation.

$$\Delta_g = \sum_{j,k=1}^n |g|^{-1/2} \frac{\partial}{\partial x^j} \left(|g|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right),$$

where $[g_{jk}]_{j,k=1}^n = g(x)$, $|g| = \det(g_{jk})$ and $[g^{jk}]_{j,k=1}^n = g(x)^{-1}$. We assume that our source W is a realization of a Gaussian white noise random variable on \mathbb{R}^{1+n} . Moreover, χ stands for a smooth function

$$\chi(t, x) = \chi_0(t) \kappa(x),$$

such that $\chi_0 \in C^\infty(\mathbb{R})$ and

$$\chi_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1, \end{cases}$$

and $\kappa \in C_0^\infty(\mathbb{R}^n)$. We assume that there exists an open and non-empty set $\mathcal{X} \subset \mathbb{R}^n$ where κ is non-vanishing. The source χW can be modelled as a random variable taking values in a local Sobolev space with negative index, and the same is true for the solution u . Contrary to papers such as [36, 39, 10], we do not aim for a random process solution of $t \mapsto u(t, \cdot)$.

The problem we study is the following: suppose we can record the empirical correlation

$$(2) \quad C_T(t_1, x_1, t_2, x_2) = \frac{1}{T} \int_0^T u(t_1 + s, x_1) u(t_2 + s, x_2) ds,$$

for $t_1, t_2 > 0$, $x_1, x_2 \in \mathcal{X}$ and $T > 0$. What information does this data yield regarding the metric g ? For any finite T , the correlation C_T is random in the sense that it depends on the realization of the source. A fundamental part of our result below is to

show that this data becomes *statistically stable*, i.e. independent of the realization, as T increases. More precisely, we show that the limit

$$\lim_{T \rightarrow \infty} \langle C_T, f \otimes h \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R}^{2+2n})}, \quad f, g \in C_0^\infty(\mathbb{R}^{1+n}),$$

is deterministic, see Theorem 3 below. Thereafter, the paper is devoted to showing that this stability enables the recovery of g :

Theorem 1. *Let $n \geq 3$. Suppose that g is non-trapping and that g coincides with the Euclidean metric outside a compact set. Let $u = \mathbb{U}(\omega)$ be the solution of (1) where $W = \mathbb{W}(\omega)$ is a realization of the white noise \mathbb{W} on \mathbb{R}^{1+n} . Then with probability one, the empirical correlations (2) defined in the sense of generalized random variables in $\mathcal{D}'((\mathbb{R} \times \mathcal{X})^2)$ for $T > 0$, determine the Riemannian manifold (\mathbb{R}^n, g) up to an isometry.*

Recall that a metric tensor g on \mathbb{R}^n is non-trapping if for each compact $K \subset \mathbb{R}^n$ there exists $T > 0$ such that for each $(p, \xi) \in T\mathbb{R}^n$, $p \in K$, $\|\xi\|_g = 1$, it holds that $\gamma_{p,\xi}(t) \notin K$ when $t \geq T$. Here we denote by $\gamma_{p,\xi}$ the unique maximal geodesic of metric g that satisfies the following initial conditions

$$\gamma_{p,\xi}(0) = x \text{ and } \dot{\gamma}_{p,\xi}(0) = \xi.$$

Note that the covariance data (2) is determined by the measurement $u|_{(0,\infty) \times \mathcal{X}}$. This implies the following corollary:

Corollary 1. *The measurement $u|_{(0,\infty) \times \mathcal{X}}$, with a single realization of the white noise source, determines the Riemannian manifold (\mathbb{R}^n, g) , up to an isometry, with probability one under the assumptions of Theorem 1.*

The statistical stability of C_T , $T > 0$, allows us to reduce the passive imaging problem to a deterministic inverse problem, that we then solve. As this deterministic problem is of independent interest, we solve it a more general geometric setting formulated as follows:

Theorem 2. *Let (N, g) be a smooth and complete Riemannian manifold of dimension $n \geq 2$. Let $\mathcal{X} \subset N$ be an open set. Consider the following initial value problem for the wave equation*

$$(3) \quad \begin{aligned} \partial_t^2 w(t, x) - \Delta_g w(t, x) &= f, \quad \text{in } (0, \infty) \times N, \\ w|_{t=0} &= \partial_t w|_{t=0} = 0. \end{aligned}$$

Let $\Lambda_{\mathcal{X}} : C_0^\infty((0, \infty) \times \mathcal{X}) \rightarrow C^\infty((0, \infty) \times \mathcal{X})$ be the local source-to-solution operator defined by

$$\Lambda_{\mathcal{X}} f = w|_{(0,\infty) \times \mathcal{X}}.$$

Then the data $(\mathcal{X}, \Lambda_{\mathcal{X}})$ determines (N, g) up to an isometry. More precisely this means the following:

Let (N_i, g_i) , $i = 1, 2$, be a smooth and complete Riemannian manifold. Let $\mathcal{X}_i \subset N_i$ be an open set, and assume that there exists a diffeomorphism

$$(4) \quad \phi : \mathcal{X}_1 \rightarrow \mathcal{X}_2$$

that satisfies

$$\phi^*(\Lambda_{\mathcal{X}_2} f) = \Lambda_{\mathcal{X}_1}(\phi^* f), \quad \text{for all } f \in C_0^\infty((0, \infty) \times \mathcal{X}_2).$$

Then there exists a Riemannian isometry $\Psi : (N_1, g_1) \rightarrow (N_2, g_2)$ such that $\Psi|_{\mathcal{X}_1} = \phi$.

Above the pullback ϕ^* of ϕ is defined by $\phi^* f = f \circ \tilde{\phi}$, where $\tilde{\phi}$ is the lift of ϕ on $(0, \infty) \times \mathcal{X}_1$, that is, $\tilde{\phi}(t, x) = (t, \phi(x))$. Note that the theorem implies that, although ϕ is apriori assumed to be only a diffeomorphism, it is in fact an isometry when it intertwines the local source-to-solution operators.

1.1. Outline the paper. We begin by showing that the empirical correlation C_T is well-defined in Section 2. In Section 3 we show the statistical stability discussed above, and in Section 4 we reduce the proof of Theorem 1 to that of Theorem 2. We prove Theorem 2 in Section 5. For the convenience of the reader, we have collected some well-known results in an appendix.

1.2. Previous literature. For previous mathematical results on passive imaging problems we refer to [17, 12] and, in particular, to the monograph [18]. Passive imaging problems arise in geophysical applications. In seismic imaging one can utilize so-called ambient seismic noise sources that appear due to nonlinear interaction of ocean waves with the ocean bottom (see e.g. [41, 42, 51]) to image the wave speed in the subsurface of the Earth.

We also mention the closely related applications of imaging random media by time reversal techniques [8, 7, 2, 14] as well as inverse scattering from random potential or random boundary conditions [9, 33, 22].

Let us now turn to results on deterministic inverse problems similar to Theorem 2. In such coefficient determination problems, it is typical to use the Dirichlet-to-Neumann map to model the data. Apart from immediate applications, this is reasonable since several other types of data can be reduced to the Dirichlet-to-Neumann case. For instance, in [37] an inverse scattering problem is solved via a reduction to the inverse conductivity problem in [43], and the latter uses the Dirichlet-to-Neumann map as data. In the present paper, however, we do not perform a reduction to the Dirichlet-to-Neumann case but adapt techniques originally developed in that case to the case of local source-to-solution map $\Lambda_{\mathcal{X}}$.

The approach that we use is a modification of the Boundary Control method. This method was first developed by Belishev to the acoustic wave equation on \mathbb{R}^n with an isotropic wave speed [4]. A geometric version of the method, suitable when the wave

speed is given by a Riemannian metric tensor as in the present paper, was introduced by Belishev and Kurylev [5]. We refer to [28] for a thorough review of the related literature. Local reconstruction of the geometry from the local source-to-solution map Λ_χ has been studied as a part of iterative schemes, see e.g. [25, 31]. In the present paper we give a global uniqueness proof that does not rely on an iterative scheme.

We restrict our attention to the unique solvability of the inverse problem but note that several variants of the Boundary Control method have been studied computationally [3, 13, 26, 40] and stability questions have been investigated [1, 30, 35].

This work continues the line of research started by the authors in [20, 21], where similar unique solvability of the geometry was considered for random and pseudo-random boundary sources. A novel feature of this paper is that we consider passive imaging, when the source is not assumed to be known.

2. THE STOCHASTIC DIRECT PROBLEM

In this section we show that the running averages C_T , $T > 0$, are well-defined. Let us first recall the concept of generalized Gaussian random variable [19]. A cylindrical set in a locally convex vector space X with the dual X' is of the form

$$\{u \in X \mid (\langle \ell_1, u \rangle, \dots, \langle \ell_k, u \rangle) \in B\},$$

where $k \geq 1$, $\ell_1, \dots, \ell_k \in X'$, and B is a Borel subset of \mathbb{R}^k , i.e., $B \in \mathcal{B}(\mathbb{R}^k)$. The σ -algebra generated by cylindrical sets in X is denoted by $\mathcal{B}_c(X)$. Notice that the cylindrical σ -algebra is always a subset of the Borel σ -algebra. However, the two σ -algebras are known to coincide if X is a separable Fréchet space [6, Thm. A.3.7].

Here, we denote the rapidly decaying functions on \mathbb{R}^n by $\mathcal{S}(\mathbb{R}^d)$. The topological dual of $\mathcal{S}(\mathbb{R}^d)$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. It is well-known that $\mathcal{S}'(\mathbb{R}^d)$ is a locally convex topological vector space (even nuclear).

Throughout the paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ stand for a complete probability space.

Definition 1. *A generalized random variable is a measurable function*

$$X : (\Omega, \mathcal{F}) \rightarrow (\mathcal{S}'(\mathbb{R}^d), \mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d))).$$

A generalized random variable X is called Gaussian, if for all $\phi_1, \dots, \phi_k \in \mathcal{S}(\mathbb{R}^d)$, $k \in \mathbb{N}$, the mapping

$$\Omega \ni \omega \mapsto (\langle X(\omega), \phi_1 \rangle, \dots, \langle X(\omega), \phi_k \rangle) \in \mathbb{R}^k$$

is a Gaussian random variable.

The probability law of a generalized Gaussian random variable X is determined by the expectation $\mathbb{E}X$ and the covariance operator $C_X : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ defined by

$$\langle \psi_1, C_X \psi_2 \rangle = \mathbb{E}(\langle X - \mathbb{E}X, \psi_1 \rangle \langle X - \mathbb{E}X, \psi_2 \rangle).$$

If X is zero-mean and satisfies $C_X = \iota$, where $\iota : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is the identity operator $\iota(\phi) = \phi$, then X is called Gaussian white noise.

Remark 1. *The construction above is identical for generalized random variables obtaining values in the space of generalized functions $\mathcal{D}'(\mathbb{R}^d)$. This was also the original formulation in [19].*

It was proved by Kusuoka in [32] that for any $\epsilon > 0$, white noise satisfies

$$\mathbb{W} \in H^{-d/2-\epsilon}(\mathbb{R}^d; \langle x \rangle^{-d/2-\epsilon}) \quad \text{almost surely,}$$

where the weight function is defined by $\langle x \rangle = (1 + |x|^2)^{1/2}$. Since the weighted Sobolev space is separable, the random variable \mathbb{W} is Radon.

It is also known that $H^{-d/2-\epsilon}(\mathbb{R}^d; \langle x \rangle^{-d/2-\epsilon}) \in \mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d))$ (see e.g. [16, Prop. 7]) and therefore we can consider \mathbb{W} as a random variable restricted to the Sobolev space $H^{-d/2-\epsilon}(\mathbb{R}^d; \langle x \rangle^{-d/2-\epsilon})$ assigned with the Borel σ -algebra. Finally, since we have a continuous embedding $H^{-d/2-\epsilon}(\mathbb{R}^d; \langle x \rangle^{-d/2-\epsilon}) \subset H_{loc}^{-d/2-\epsilon}(\mathbb{R}^d)$, we identify \mathbb{W} as a random variable

$$\mathbb{W} : (\Omega, \mathcal{F}) \rightarrow (H_{loc}^{-d/2-\epsilon}(\mathbb{R}^d), \mathcal{B}(H_{loc}^{-d/2-\epsilon}(\mathbb{R}^d)))$$

Notice now that the Radon property is transferred through any continuous mapping between locally convex spaces [6]. It follows that the random variable \mathbb{W} (and later also C_T) is Radon.

We denote by \square_χ^{-1} the solution operator of (1), that is, $\square_\chi^{-1}(W) = u$ where u solves (1) and u is defined to be zero for negative times. Then

$$\square_\chi^{-1} : Z \rightarrow H_{loc}^{\sigma+1}(\mathbb{R}^{1+n}), \quad \sigma \in \mathbb{R},$$

is continuous, see e.g. [24, Thm. 23.2.4], where

$$Z = \{f \in H_{loc}^\sigma(\mathbb{R}^{1+n}) : \text{supp}(f) \subset \overline{\mathbb{R}_+} \times \mathbb{R}^n\}.$$

We denote by τ^s the translation by $s \in \mathbb{R}$ in time, that is,

$$\tau^s \phi(t) = \phi(t + s), \quad \phi \in C_0^\infty(\mathbb{R}),$$

and extend this definition to $\mathcal{D}'(\mathbb{R})$ by

$$\langle \tau^s w, \phi \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R})} = \langle w, \tau^{-s} \phi \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R})}.$$

The function

$$\Phi : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}, \quad \Phi(s, t) = \tau^s \phi(t)$$

is smooth, and moreover $\Phi = 0$ when $t \notin (0, T + R)$ where $R > 0$ is such that $\text{supp}(\phi) \subset (0, R)$. Hence function

$$s \mapsto \langle w, \Phi(s, \cdot) \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R})} = \langle \tau^s w, \phi \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R})}$$

is smooth for all $w \in \mathcal{D}'(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$, see [23, Thm. 2.1.3]. An analogous argument shows that

$$s \mapsto \langle \tau^s w \otimes \tau^s w, \phi \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R}^{2+2n})}$$

is smooth for all $w \in \mathcal{D}'(\mathbb{R}^{1+n})$ and $\phi \in C_0^\infty(\mathbb{R}^{2+2n})$. Here \otimes denotes the tensor product of distributions, see e.g. [23, Thm. 5.1.1] for the definition.

For a fixed $T > 0$, we define the map

$$A_T(w) = \frac{1}{T} \int_0^T \tau^s w \otimes \tau^s w \, ds, \quad w \in H_{loc}^\sigma(\mathbb{R}^{1+n}),$$

in the sense of the Pettis integral, that is,

$$\langle A_T(w), \phi \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R}^{2+2n})} = \frac{1}{T} \int_0^T \langle \tau^s w \otimes \tau^s w, \phi \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R}^{2+2n})} \, ds.$$

The integral above defines $A_T(w)$ as a generalized function in $\mathcal{D}'(\mathbb{R}^{2+2n})$ and, moreover, yields a continuous map in the following sense:

Lemma 1. *The map $A_T : H_{loc}^{-\sigma}(\mathbb{R}^{1+n}) \rightarrow H_{loc}^{-\sigma}(\mathbb{R}^{2+2n})$, $\sigma \in \mathbb{R}$, is continuous.*

Proof. We recall that the topology of $H_{loc}^{-\sigma}(\mathbb{R}^{1+n})$ is induced by the semi-norms

$$w \mapsto \|\psi w\|_{H^{-\sigma}(\mathbb{R}^{1+n})}, \quad \psi \in C_0^\infty(\mathbb{R}^{1+n}).$$

Let $w_0 \in H_{loc}^{-\sigma}(\mathbb{R}^{1+n})$, $\psi \in C_0^\infty(\mathbb{R}^{2+2n})$ and $\epsilon > 0$. In order to show that A_T is continuous, it is enough to show [46, p. 64] that there are $\tilde{\psi} \in C_0^\infty(\mathbb{R}^{1+n})$ and $\delta > 0$ such that

$$\left\| \tilde{\psi}(w - w_0) \right\|_{H^{-\sigma}(\mathbb{R}^{1+n})} < \delta \quad \text{implies} \quad \left\| \psi(A_T(w) - A_T(w_0)) \right\|_{H^{-\sigma}(\mathbb{R}^{2+2n})} < \epsilon.$$

We choose $\tilde{\psi} \in C_0^\infty(\mathbb{R}^{1+n})$ so that $(\tilde{\psi} \otimes \tilde{\psi})\tau_1^{-s}\tau_2^{-s}\psi = \tau_1^{-s}\tau_2^{-s}\psi$ for all $s \in (0, T)$. Here τ_j^{-s} , $j = 1, 2$, act in the different time variables. Let $\phi \in H^\sigma(\mathbb{R}^{2+2n})$. It follows that

$$\begin{aligned} & |\langle \psi(A_T(w) - A_T(w_0)), \phi \rangle_{H^{-\sigma} \times H^\sigma(\mathbb{R}^{2+2n})}| \\ & \leq \frac{1}{T} \int_0^T \left\| (\tilde{\psi} \otimes \tilde{\psi})(w \otimes w - w_0 \otimes w_0) \right\|_{H^{-\sigma}(\mathbb{R}^{2+2n})} \left\| \tau_1^{-s}\tau_2^{-s}\psi\phi \right\|_{H^\sigma(\mathbb{R}^{2+2n})} \, ds \\ & \leq C \left\| \tilde{\psi}(w - w_0) \otimes \tilde{\psi}w + \tilde{\psi}w_0 \otimes \tilde{\psi}(w - w_0) \right\|_{H^{-\sigma}(\mathbb{R}^{2+2n})} \left\| \phi \right\|_{H^\sigma(\mathbb{R}^{2+2n})}. \end{aligned}$$

Finally, for small $\delta > 0$

$$\begin{aligned} & \left\| \tilde{\psi}(w - w_0) \otimes \tilde{\psi}w + \tilde{\psi}w_0 \otimes \tilde{\psi}(w - w_0) \right\|_{H^{-\sigma}(\mathbb{R}^{2+2n})} \\ & \leq \delta \left\| \tilde{\psi}w \right\|_{H^{-\sigma}(\mathbb{R}^{1+n})} + \left\| \tilde{\psi}w_0 \right\|_{H^{-\sigma}(\mathbb{R}^{1+n})} \delta \leq C\delta. \end{aligned}$$

□

We define $C_T(w) = A_T(\square_\chi^{-1}(\mathbb{W}(w)))$, $T > 0$, and see that

$$C_T : \Omega \rightarrow H_{loc}^\sigma(\mathbb{R}^{2+2n}), \quad \sigma < -\frac{1+n}{2} + 1,$$

is a random variable. As mentioned above, the Radon property is transferred through continuous mappings and therefore C_T is Radon.

3. THE STOCHASTIC INVERSE PROBLEM AND STATISTICAL STABILITY

For any function $f \in C_0^\infty(\mathbb{R}^{1+n})$, let us define $v^f = v$ as the solution of a time reversed wave equation

$$(5) \quad \begin{aligned} \partial_t^2 v - \Delta_g v &= f \quad \text{in } (-\infty, S) \times \mathbb{R}^n, \\ v|_{t=S} &= \partial_t v|_{t=S} = 0, \end{aligned}$$

where $S \in \mathbb{R}$ is large enough so that $f \in C_0^\infty((-\infty, S) \times \mathbb{R}^n)$. In this section we show the following theorem.

Theorem 3. *Suppose that $n \geq 3$, (\mathbb{R}^n, g) is non-trapping and that g coincides with the Euclidean metric outside a compact set. Let $\mathbb{D} \subset C_0^\infty((0, \infty) \times \mathcal{X})$ be a countable set. There exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 0$ and for all $\omega \in \Omega \setminus \Omega_0$ and all $f, h \in \mathbb{D}$, it holds that*

$$\lim_{T \rightarrow \infty} \langle C_T(\omega), f \otimes h \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R}^{2+2n})} = \langle \kappa v^f, \kappa v^h \rangle_{L^2(\mathbb{R}^{1+n})}.$$

In what follows, we write $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R}^{2+2n})}$ for the distributional duality but include the subscript whenever convenient.

Lemma 2. *Let $W \in \mathcal{D}'(\mathbb{R}^{1+n})$ and $f \in C_0^\infty(\mathbb{R}^{1+n})$ be arbitrary sources in problems (1) and (5), respectively. Moreover, let u and v^f be the corresponding solutions. Then we have the identity*

$$\langle u, f \rangle = \langle W, \chi v^f \rangle.$$

Proof. Suppose that $W \in C_0^\infty(\mathbb{R}^{1+n})$. The general case follows since test functions are dense in distributions. Next, let v and S be as in (5). Using the shorthand notation $\square = \partial_t^2 - \Delta_g$, we have that

$$\langle u, f \rangle = \langle u, \square v \rangle_{L^2((0, S) \times \mathbb{R}^n)} = \langle \square u, v \rangle_{L^2((0, S) \times \mathbb{R}^n)} = \langle W, \chi v \rangle.$$

This proves the claim. \square

Let us recall the following well-known result regarding the local energy decay which is due to Vainberg [48, 47], see [49] for the formulation as below.

Theorem 4. *Let $u \in C^\infty((0, \infty) \times \mathbb{R}^n)$ solve the problem*

$$\begin{aligned} \partial_t^2 u - \Delta_g u &= 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n, \\ u|_{t=0} &= u_0, \quad \partial_t u|_{t=0} = u_1. \end{aligned}$$

Suppose that u_0 and u_1 are compactly supported. Suppose that (\mathbb{R}^n, g) is non-trapping and that g coincides with the Euclidean metric outside a compact set. Then there is $t_0 > 0$ such that u satisfies local energy decay

$$\int_{\mathbb{R}^n} (|\partial_t u(t, x)|^2 + |\nabla_g u(t, x)|^2) \chi(x) dV_g(x) \leq C\eta(t)E_0, \quad t > t_0,$$

for any compactly supported function $\chi \in C_0^\infty(\mathbb{R}^n)$. Here ∇_g stands for the gradient of metric tensor g . Here

$$E_0 = \int_{\mathbb{R}^n} |\nabla_g u_0(x)|^2 + |u_1(x)|^2 dV_g(x), \quad \eta(t) = \begin{cases} e^{-bt}, & n \geq 3 \text{ odd}, \\ t^{-2n}, & n \geq 2 \text{ even}, \end{cases}$$

and the constants $C, b > 0$ depend on n, g, χ and the supports of u_0 and u_1 .

We need a decay estimate for the norm $\|u(t, \cdot)\|_{L^2(K)}$ where $K \subset \mathbb{R}^n$ is compact.

Lemma 3. *Let (\mathbb{R}^n, g) be as in Theorem 3 and let u be as in Theorem 4. Let $K \subset \mathbb{R}^n$ be compact. Then there is $t_0 > 0$ such that u satisfies*

$$\|u(t, \cdot)\|_{L^2(K)} \leq C\mu(t)E_0, \quad t > t_0,$$

where

$$(6) \quad \mu(t) = \begin{cases} e^{-bt}, & n \geq 3 \text{ odd}, \\ t^{-2n+1}, & n \geq 4 \text{ even}, \end{cases}$$

Proof. To simplify the notation, we assume without loss of generality that $E_0 = 1$. We point out that all the integrals in the proof will be Euclidean. Let $B(r) = \{\|x\| < r\}$ be the Euclidean ball of radius r and write

$$u_r(t) = \frac{1}{|B(r)|} \int_{B(r)} u(t, x) dx.$$

Theorem 4 implies $|\partial_t u_r(t)| \leq C\eta(t)$ where the constant $C > 0$ depends on $r > 0$ and g . Thus for $t_0 < t < s$,

$$(7) \quad |u_r(t) - u_r(s)| \leq C \int_t^s \eta(\tau) d\tau = C(\mu(t) - \mu(s)).$$

We see that $\lim_{t \rightarrow \infty} u_r(t)$ exists, and denote the limit by $\bar{u}(r)$.

The Poincaré-Wirtinger inequality

$$\|u(t, \cdot) - u_r(t)\|_{L^2(B(r))} \leq C \|\nabla u(t, \cdot)\|_{L^2(B(r))},$$

together with Theorem 4, g is Euclidean metric outside compact set, and (7), implies that

$$(8) \quad \|u(t, \cdot) - \bar{u}(r)\|_{L^2(B(r))} \leq C\eta(t) + |u_r(t) - \bar{u}(r)| \|1\|_{L^2(B(r))} \leq C\mu(t).$$

In particular, for $0 < r_1 < r_2$, $u(t, \cdot) \rightarrow \bar{u}(r_j)$, $j = 1, 2$, in $L^2(B(r_1))$. Thus $\bar{u}(r)$ does not depend on $r > 0$ and we denote it by \bar{u} .

It remains to show that $\bar{u} = 0$. As $u(t)$ is compactly supported, by the finite speed of propagation, the Gagliardo-Nirenberg-Sobolev inequality implies that

$$\|u(t, \cdot)\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)},$$

where p^* is the Sobolev conjugate of 2, that is, $1/p^* = 1/2 - 1/n$. Note that $p^* > 2$. We apply Hölder's inequality with $p = p^*/2$ and $1/p + 1/q = 1$,

$$\int_{B(r)} u^2(t, \cdot) dx \leq \|u^2(t, \cdot)\|_{L^p(B(r))} \|1\|_{L^q(B(r))}.$$

The conservation of energy implies that $\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)}$, $t > 0$, is bounded. Thus $\|u(t, \cdot)\|_{L^2(B(r))}^2 \leq Cr^{n/q}$ with a constant $C > 0$ independent of r .

To get a contradiction, suppose now that $\bar{u} \neq 0$. Then there is $\epsilon > 0$ such that

$$\|\bar{u}\|_{L^2(B(r))}^2 = \bar{u}^2 \|1\|_{L^2(B(r))}^2 = 2\epsilon r^n.$$

By the convergence (8), for all $r > 0$ there is t_r such that $\|u(t_r, \cdot)\|_{L^2(B(r))}^2 \geq \epsilon r^n$. Thus $r^{n-n/q} \leq C$, $r > 0$, which is a contradiction since $q > 1$. \square

Lemma 4. *Let (\mathbb{R}^n, g) be as in Theorem 3. Suppose that $K \subset \mathbb{R}^n$ is compact and $f \in C_0^\infty(\mathbb{R}^n)$. Let $u \in C^\infty((0, \infty) \times \mathbb{R}^n)$ solve the problem*

$$\begin{aligned} \partial_t^2 u - \Delta_g u &= f, \quad \text{in } (0, \infty) \times \mathbb{R}^n, \\ u|_{t=0} &= \partial_t u|_{t=0} = 0. \end{aligned}$$

Then there exists $t_0 > 0$ such that for all $t > t_0$

$$\|u(t, \cdot)\|_{L^2(K)} \leq C\mu(t)\|f\|_{L^2(\mathbb{R}^{1+n})},$$

where $\mu(t)$ is defined in (6). Here the constants C and t_0 depend on g , K and the support of f .

Proof. Let $t_1 > 0$ be such that $\text{supp}(f) \subset [0, T] \times \mathbb{R}^n$. By the finite speed of wave propagation, it holds that $\text{supp}(u|_{t=t_1})$ and $\text{supp}(\partial_t u|_{t=t_1})$ are compact in \mathbb{R}^n . Consider the solution v of the initial value problem

$$\begin{aligned} \partial_t^2 v - \Delta_g v &= 0, \quad \text{in } (t_1, \infty) \times \mathbb{R}^n, \\ v|_{t=t_1} &= u(t_1), \quad \partial_t v|_{t=t_1} = \partial_t u|_{t=t_1}. \end{aligned}$$

By the uniqueness, it must hold that $v = u$. By Lemma 3 there exists $t_0 > t_1$ and constant C independent of $t > t_0$ such that

$$\|u(t, \cdot)\|_{L^2(K)} \leq C\mu(t)E_0, \quad t > t_0,$$

Where $E_0 = \int_{\mathbb{R}^n} |\nabla_g u(t_1, \cdot)|^2 + |\partial_t u(t_1, \cdot)|^2 dV_g(x)$. As u is an energy class solution of a wave equation with zero initial values, by standard Energy estimates it holds that

$$E_0 \leq C\|f\|_{L^2(\mathbb{R}^{1+n})}.$$

This proves the claim. \square

Lemma 5. *Let (\mathbb{R}^n, g) be as in Theorem 3. Let $f, h \in C_0^\infty((0, S) \times \mathbb{R}^n)$. It follows that*

$$\lim_{T \rightarrow \infty} \mathbb{E}\langle C_T, f \otimes h \rangle = \langle \kappa v^f, \kappa v^h \rangle_{L^2((-\infty, S) \times \mathbb{R}^n)}.$$

Proof. Here we will use notation $f^s(t, x) = f(t + s, x)$ for a time shift $s \in \mathbb{R}$. By the Lemma 2 and standard energy estimates, we have

$$\mathbb{E}\langle u^s, f \rangle^2 = \mathbb{E}\langle \mathbb{W}^s, \chi^s v^f \rangle^2 = \langle \chi^s v^f, v^f \rangle \leq C(T) \|f\|_{L^2(\mathbb{R}^{1+n})}^2$$

therefore, using Young's inequality we see that the mapping

$$(\omega, s) \rightarrow \langle u^s(\omega), f \rangle \langle u^s(\omega), g \rangle$$

is integrable on $\Omega \times (0, T)$ for any fixed $T > 0$ with respect to $\mathbb{P} \times dt$. In consequence, applying the Fubini theorem now yields

$$\begin{aligned} \mathbb{E}\langle C_T, f \otimes h \rangle &= \frac{1}{T} \mathbb{E} \int_0^T \langle u^s, f \rangle \langle u^s, g \rangle ds \\ &= \frac{1}{T} \int_0^T \mathbb{E} \langle \chi^s \mathbb{W}^s, v^f \rangle \langle \chi^s \mathbb{W}^s, v^h \rangle ds \\ &= \frac{1}{T} \int_0^T \langle \chi^s v^f, \chi^s v^h \rangle ds. \end{aligned}$$

For the time-shifted characteristic function we have

$$\chi^s(t, x) = \chi_0^s(t) \kappa(x) = \kappa(x) - (1 - \chi_0^s(t)) \kappa(x)$$

and $\text{supp}(1 - \chi_0^s) \subset (-\infty, 1 - s)$. By the local energy decay in Lemma 4 there is $C > 0$ depending on n, g , and the supports of κ and f such that

$$\|v^f\|_{L^2((-\infty, 1-s) \times \text{supp}(\kappa))} \leq C \left(\int_{-\infty}^{1-s} |t|^{-4n+2} dt \right)^{1/2} \|f\|_{L^2(\mathbb{R}^{1+n})} \leq C s^{-2n+\frac{3}{2}} \|f\|_{L^2(\mathbb{R}^{1+n})},$$

for large s . Hence we obtain

$$\mathbb{E}\langle C_T, f \otimes h \rangle = \langle \kappa v^f, \kappa v^h \rangle + \frac{1}{T} \int_0^T R(s) ds,$$

where

$$|R(s)| \leq C s^{-2n+\frac{3}{2}} \|f\|_{L^2(\mathbb{R}^{1+n})} \|h\|_{L^2(\mathbb{R}^{1+n})}$$

To conclude, one has

$$\frac{1}{T} \int_0^T |R(s)| ds \leq C T^{-2n+\frac{3}{2}}$$

and the claim follows. \square

In order to show statistical stability of the data, we need some basic results from ergodic theory. The next theorem given in [11, p. 94] provides a suitable condition.

Theorem 5. *Let \tilde{Z}_t with $t \geq 0$ be a real-valued stochastic process with continuous paths and zero-mean $\mathbb{E}\tilde{Z}_t = 0$. Assume that for some constants $c, \epsilon > 0$ the condition*

$$|\mathbb{E}(\tilde{Z}_t \tilde{Z}_{t+r})| \leq c(1+r)^{-\epsilon}$$

holds for all $t \geq 0$ and $r \geq 0$. Then,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{Z}_t dt = 0 \quad \text{almost surely.}$$

Lemma 6. *Let (\mathbb{R}^n, g) be as in Theorem 3. Let $f, h \in C_0^\infty((0, S) \times \mathbb{R}^n)$ and use notation*

$$Z_r = \langle u^r, f \rangle \langle u^r, h \rangle$$

Then there is $C > 0$ depending on n, g , and the supports of κ, f and h such that

$$|\mathbb{E}(Z_r - \mathbb{E}Z_r)(Z_{r+s} - \mathbb{E}Z_{r+s})| \leq C(1+s)^{-n} \|f\|_{L^2(\mathbb{R}^{1+n})}^2 \|h\|_{L^2(\mathbb{R}^{1+n})}^2.$$

Proof. For convenience, let us write $X^r = \langle u^r, f \rangle$ and $Y^r = \langle u^r, h \rangle$. By the well-known Isserlis's formula for Gaussian random variables we have

$$\mathbb{E}X^r Y^r X^{r+s} Y^{r+s} = \mathbb{E}X^r Y^r \mathbb{E}X^{r+s} Y^{r+s} + \mathbb{E}X^r X^{r+s} \mathbb{E}Y^r Y^{r+s} + \mathbb{E}X^r Y^{r+s} \mathbb{E}Y^r X^{r+s}$$

and, consequently,

$$(9) \quad \mathbb{E}(Z_r - \mathbb{E}Z_r)(Z_{r+s} - \mathbb{E}Z_{r+s}) = \mathbb{E}X^r X^{r+s} \mathbb{E}Y^r Y^{r+s} + \mathbb{E}X^r Y^{r+s} \mathbb{E}Y^r X^{r+s}.$$

We write $v_r^f(t, \cdot) = v^f(t-r, \cdot)$. The local energy decay, Lemma 4, implies

$$\begin{aligned} |\mathbb{E}X^r Y^{r+s}| &= |\mathbb{E}\langle \chi^r \mathbb{W}^r, v^f \rangle \langle \chi^{r+s} \mathbb{W}^{r+s}, v^h \rangle| \\ &= |\mathbb{E}\langle \chi \mathbb{W}, v_r^f \rangle \langle \chi \mathbb{W}, v_{r+s}^h \rangle| \\ &= |\langle \chi v_r^f, \chi v_{r+s}^h \rangle| \\ (10) \quad &\leq C(1+s)^{-2n+\frac{3}{2}} \|f\|_{L^2(\mathbb{R}^{1+n})} \|h\|_{L^2(\mathbb{R}^{1+n})}, \end{aligned}$$

since v_r^f is small in $\text{supp}(v_{r+s}^h)$ for $s \gg 0$. \square

Proof of Theorem 3. The claim follows for a fixed pair of sources (f, h) by setting $\tilde{Z}_t = Z_t - \mathbb{E}Z_t$ and combining results from Lemma 5, Theorem 5 and Lemma 6. We then proceed by repeating the argument simultaneously for a countable set of source pairs (countable union of zero measurable sets is zero measurable). \square

We conclude this section with the following simple lemma to quantify the convergence of the data. Notice that Lemma 7 is not needed for the previous proof.

Lemma 7. *Let $f, h \in C_0^\infty((0, S) \times \mathbb{R}^n)$. Then there is $C > 0$ depending on n, g , and the supports of κ, f and h such that*

$$\text{Var}\langle C_T, f \otimes h \rangle \leq CT^{-2} \|f\|_{L^2(\mathbb{R}^{1+n})}^2 \|h\|_{L^2(\mathbb{R}^{1+n})}^2$$

Proof. In the proof of Lemma 5 we showed that the Gaussian random variables X^r and Y^r have a bounded variance independent of r . Since any moment of a Gaussian random variable is bounded by a constant depending on the variance, we see that the mapping

$$(\omega, r, s) \rightarrow X^r Y^r X^s Y^s$$

is integrable over $\Omega \times (0, T) \times (0, T)$ for any fixed $T > 0$ with respect to $\mathbb{P} \times dr \times ds$.

Now the Fubini theorem yields that

$$\begin{aligned} \mathbb{E}\langle C_T, f \otimes h \rangle^2 &= \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} X^s Y^s X^r Y^r ds dr \quad \text{and} \\ (\mathbb{E}\langle C_T, f \otimes h \rangle)^2 &= \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} X^s Y^s \mathbb{E} X^r Y^r ds dr. \end{aligned}$$

It follows by equation (9) and estimate (10) that

$$\text{Var}(\langle C_T, f \otimes h \rangle) \leq C \|f\|_{L^2(\mathbb{R}^{1+n})}^2 \|h\|_{L^2(\mathbb{R}^{1+n})}^2 \frac{1}{T^2} \int_0^T \int_0^T (1 + |r - s|)^{-4n+3} ds dr$$

and the claim follows by estimating the the double integral in time by

$$\begin{aligned} \int_0^T \int_0^T (1 + |r - s|)^{-4n+3} ds dr &= \frac{1}{2} \int_0^{2T} \int_{2T-s'}^{s'} (1 + r')^{-4n+4} dr' ds' \\ &= \frac{1}{2(1-n)} \int_0^{2T} ((1 + s')^{1-n} - (1 + 2T - s')^{1-n}) ds' \\ &\leq C(1 + T^{5-4n}) \leq C. \end{aligned}$$

for any $n \geq 3$. \square

4. REDUCTION TO THE DETERMINISTIC INVERSE PROBLEM

We will provide the reconstruction method for g in a more general setting: (N, g) is a smooth complete Riemannian manifold with out boundary and $\kappa \in C_0^\infty(N)$. Let $\mathcal{X} \subset N$ be open and bounded set such that $\kappa|_{\mathcal{X}}$ is non-vanishing. We will assume that we know the smooth structure of \mathcal{X} . By replacing \mathcal{X} with a smaller set we may assume without loss of generality that $\partial\mathcal{X}$ is smooth. To summarize, we assume that \mathcal{X} is known as a smooth open manifold with smooth boundary.

For a function $f \in C_0^\infty((0, \infty) \times N)$, we will use the notation $w^f = w \in C^\infty(\mathbb{R} \times N)$ for the unique solution of the wave equation

$$(11) \quad \begin{aligned} \partial_t^2 w - \Delta_g w &= f \quad \text{in } (0, \infty) \times N, \\ w|_{t=0} &= \partial_t w|_{t=0} = 0. \end{aligned}$$

We will often use a shorthand notation \square_g for the wave operator $\partial_t^2 w - \Delta_g$. For a function $f : \mathbb{R} \times N \rightarrow \mathbb{R}$ and $t_0 \in \mathbb{R}$ we sometimes use the shorthand notation $f(t_0) := f(t_0, \cdot)$, if the time variable is considered to be fixed.

Suppose that $\mathbb{D} \subset C_0^\infty((0, \infty) \times \mathcal{X})$ is countable and dense. We assume that the following *inner product data*

$$(12) \quad \{\mathcal{X}, I_{\mathbb{D}}\}$$

is given. Here \mathcal{X} stands for the assumption that the topological and smooth structure of \mathcal{X} is known and $I_{\mathbb{D}}$ stands for the assumption that function

$$I_{\mathbb{D}} : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{R}, \quad I_{\mathbb{D}}(f, h) = \langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)}$$

is known.

Observe that the wave equation (11) is invariant with respect to any time shift and reversion of time. Therefore knowing inner products $\langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)}$ for solutions of equation (11) is equivalent for knowing the inner products $\langle \kappa v^f, \kappa v^h \rangle_{L^2(\mathbb{R} \times N)}$ for the time reversed wave equation. Thus we can consider that the data (12) is given by Theorem 3.

In this section we will show the following theorem.

Theorem 6. *Let $\mathbb{D} \subset C_0^\infty((0, \infty) \times \mathcal{X})$ be a countable dense set. Then the inner product data (12) determine the local source-to-solution map*

$$\Lambda_{\mathcal{X}} f := w^f|_{(0, \infty) \times \mathcal{X}}, \quad f \in C_0^\infty((0, \infty) \times \mathcal{X}),$$

where w^f is a solution of (11).

It follows from the assumptions in Theorem 1 that the Riemannian manifold (\mathbb{R}^n, g) is complete. Indeed, the metric tensor g coincides with the Euclidean metric e outside a compact set, and therefore there exist uniform constants $c, C > 0$ such

that $c\|\cdot\|_e \leq \|\cdot\|_g \leq C\|\cdot\|_e$, where $\|\cdot\|_e$ stands for the Euclidean and $\|\cdot\|_g$ for the Riemannian norm. Thus Theorems 3, 6 and 2 imply Theorem 1.

We will prove two auxiliary lemmas before presenting a proof for Theorem 6. For notational convenience we choose an auxiliary Riemannian distance function $d_0 : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ that gives the topology of \mathcal{X} . For a given $p \in \mathcal{X}$ and $r > 0$, we will denote by $B_0(p, r) \subset \mathcal{X}$ the open ball of the metric space (\mathcal{X}, d_0) . We will denote the distance function given by metric tensor g as d_g .

Definition 2. Let $a < b$ and $U \subset N$ be open. Also let $\mathcal{U} = (a, b) \times U \subset \mathbb{R} \times N$. Then we say that $f \in C_0^\infty(\mathcal{U})$ is essentially non-radiating, if $\text{supp}(w^f) \subset \overline{\mathcal{U}}$.

Another concept that we will need is the *open future* of a set \mathcal{B} in a space time $\mathbb{R} \times N$.

Definition 3. We define the future of a set $\mathcal{B} \subset \mathbb{R} \times N$ by

$$\begin{aligned} \mathcal{I}^+(\mathcal{B}) = \{ & (t, x) \in \mathbb{R} \times N \mid \text{there exists } (s, y) \in \mathcal{B} \text{ such that } t > s \\ & \text{and } d_g(x, y) < t - s \}. \end{aligned}$$

Lemma 8. Let $(t_0, x_0) \in \mathbb{R} \times \mathcal{X}$ and $\epsilon > 0$. We define

$$B_\epsilon(x_0, t_0) = (t_0 - \epsilon, t_0) \times B_0(x_0, \epsilon) \subset \mathbb{R} \times \mathcal{X}.$$

For small $\epsilon > 0$ we have that $f \in C_0^\infty(B_\epsilon(x_0, t_0))$ is essentially non-radiating if and only if

$$(13) \quad \langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)} = 0$$

for all $h \in C_0^\infty(Q)$, where $Q = (t_0, t_0 + 1) \times \mathcal{X}$.

Proof. Let $\epsilon > 0$ be so small that for all $(s, q) \in B_\epsilon(x_0, t_0)$

$$(14) \quad \mathcal{I}^+(\{(s, q)\}) \cap (\{t_0\} \times N) \subset \{t_0\} \times \mathcal{X}.$$

We use a shorthand notation $B := B_0(x_0, \epsilon)$ and assume that ϵ is also so small that the set

$$G := \{x \in N : \text{dist}_g(x, B) < \epsilon/2\} \subset B_g(x_0, \delta)$$

for some $\delta \in (0, \text{inj}(x_0))$. Here $\text{inj}(x_0)$ is the injectivity radius of x_0 with respect to g . Suppose that $f \in C_0^\infty(B_\epsilon(x_0, t_0))$ is essentially non-radiating. Then it holds

$$w^f(\cdot, t) = 0, \text{ for all } t > t_0.$$

On the other hand for all $h \in C_0^\infty(Q)$ it holds that

$$\text{supp}(w^h) \subset (t_0, \infty) \times N.$$

Therefore equation (13) holds.

Let $f \in C_0^\infty(B_\epsilon(x_0, t_0))$ be such that, for the solution $w^f \in C^\infty(\mathbb{R} \times N)$ of (11), the equation (13) holds for all $h \in C_0^\infty(Q)$. Let $\phi \in C_0^\infty(Q)$. By choosing $h = \square_g(\kappa^{-2}\phi)$ we have $w^h = \kappa^{-2}\phi$ and further

$$\begin{aligned} \langle w^f, \phi \rangle_{L^2(\mathbb{R} \times N)} &= \langle \kappa^2 w^f, \kappa^{-2} \phi \rangle_{L^2(\mathbb{R} \times N)} = \langle \kappa^2 w^f, w^h \rangle_{L^2(\mathbb{R} \times N)} \\ &= \langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)} = 0. \end{aligned}$$

Thus we can deduce that $w^f|_Q \equiv 0$. By (14) and the finite speed of wave propagation, there exists $\delta > 0$ such that $\text{supp}(w^f(t, \cdot)) \subset Q$ for any $t \in [t_0, t_0 + \delta]$. Therefore by the uniqueness of solutions we have $w^f|_{(t_0, \infty) \times N} \equiv 0$. We can easily check that

$$w^f|_{((t_0, \infty) \times N) \cup ((t_0 - \epsilon, \infty) \times N)} \equiv 0.$$

Next we will investigate what happens in the strip $(t_0 - \epsilon, t_0) \times N$. Let $(s, y) \in (t_0 - \epsilon, t_0) \times N$. If $\text{dist}_g(y, B) \geq s - t_0 + \epsilon$, we can again use the finite speed of wave propagation (Thm. 9) to conclude that $w^f(y, s) = 0$. Reversing the time we also deduce that $w^f(y, s) = 0$, if $\text{dist}_g(y, B) \geq t_0 - s$. Therefore we only have to consider what happens in the set

$$C = \{(s, y) \in (t_0 - \epsilon, t_0) \times N : \text{dist}_g(B, y) < t_0 - s, \text{dist}_g(B, y) < s - t_0 + \epsilon\}.$$

We will present a cutting process that eventually shows that $\text{supp}(w^f) \subset \overline{B_\epsilon(x_0, t_0)}$. First notice that for all $(s, y) \in C$ it holds that $\text{dist}_g(y, B) < \frac{\epsilon}{2}$ and by (14) it holds $C \cap (\{s\} \times N) \subset \{s\} \times \mathcal{X}$. Let $s = t_0 - \epsilon/2$.

Recall that by the choice of ϵ we have $(C \cap (\{s\} \times N)) \subset G \subset B_g(x_0, \delta)$ and $\delta < \text{inj}(x_0)$. Therefore it holds that there exist $t_1 \in (\epsilon/2, \delta)$ and $T \in (0, t_1 - \frac{\epsilon}{2})$ for every $\xi \in S_{x_0}N$, such that

$$\begin{aligned} 2T + t_1 &< \text{inj}(x_0), \\ ((s - T, s + T) \times B_g(\gamma_{x_0, \xi}(t_1), T)) \cap C &= \emptyset, \\ ((s - T, s + T) \times B_g(\gamma_{x_0, \xi}(t_1), 2T)) \cap C &\neq \emptyset \end{aligned}$$

and

$$((s - T, s + T) \times B_g(\gamma_{x_0, \xi}(t_1), 2T)) \cap B_\epsilon(x_0, t_0) = \emptyset.$$

Therefore the following holds:

$$w^f|_{(s-T, s+T) \times B_g(\gamma_{x_0, \xi}(t_1), T)} \equiv 0 \text{ and } \square_g w^f = f = 0 \text{ in } (s - T, s + T) \times B_g(\gamma_{x_0, \xi}(t_1), 2T).$$

By Tataru's unique continuation theorem (Thm. 10), it holds that $w^f(s, y) = 0$ for any $\xi \in S_{x_0}N$ and $y \in B_g(\gamma_{x_0, \xi}(t_1), 2T)$. Since we assumed that $s = t_0 - \epsilon/2$ we conclude that

$$w^f(t, y) = 0, \text{ if } (t, y) \in S_1 := (t_0 - \epsilon, t_0) \times \left(\bigcup_{\xi \in S_{x_0}N} B(\gamma_{x_0, \xi}(t_1), 2T) \right).$$

Let $C_2 := C \setminus S_1$. Since T depends only on δ , we can iterate the previous cutting procedure to show that $\text{supp}(w^f) \subset \overline{B_\epsilon(x_0, t_0)}$. \square

We define the open *domain of influence* for a set $A \subset N$ as

$$M(s, A) := \{x \in N : d_g(x, A) < s\}.$$

Lemma 9. *Let $y_0, x_0 \in \mathcal{X}$. Let $\epsilon_1, \epsilon_2 > 0$ be so small that $E_1(\epsilon_1) := B_0(y_0, \epsilon_1) \subset \mathcal{X}$ and $E_2(\epsilon_2) := B_0(x_0, \epsilon_2) \subset \mathcal{X}$. We write $\mathcal{C} = (0, \infty) \times E_1$. Let t_0 . We also define $B_{t_0} = (-\infty, t_0) \times E_2$. Then*

$$(15) \quad \mathcal{I}^+(\mathcal{C}) \cap B_{t_0} = \emptyset,$$

if and only if for all $h \in C_0^\infty(\mathcal{C})$ and all essentially non-radiating $f \in C_0^\infty(B_{t_0})$ it holds that

$$(16) \quad \langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)} = 0.$$

Especially it holds that

$$(17) \quad \text{dist}_g(E_1(\epsilon_1), E_2(\epsilon_2)) = \sup\{t_0 > 0 : (15) \text{ is valid}\}$$

and

$$d_g(x_0, y_0) = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \text{dist}_g(E_1(\epsilon_1), E_2(\epsilon_2)).$$

Proof. Suppose that (15) is valid. Let $h \in C_0^\infty(\mathcal{C})$ and $f \in C_0^\infty(B_{t_0})$ be essentially non-radiating. Then

$$\text{supp}(w^f) \subset \overline{B_{t_0}}.$$

Since $\text{supp}(h) \subset \mathcal{C}$, it holds that $\text{supp}(w^h) \subset \mathcal{I}^+(\mathcal{C})$ by the finite speed of wave propagation. Therefore

$$\langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)} = 0.$$

Suppose (15) does not hold. We will denote the open non-empty set $\mathcal{A} := \mathcal{I}^+(\mathcal{C}) \cap B_{t_0}$. Let $\phi \in C_0^\infty(\mathcal{A})$ be non-zero and $\phi \geq 0$. Choose $(s, x) \in \mathcal{A}$ such that $\phi(s, x) \neq 0$. By approximate controllability (Theorem 11) there exists a source $h \in \mathcal{C}$ such that

$$\langle \phi(s), w^h(s) \rangle_{L^2(N)} > 0.$$

This holds only, if there exists a point $(s, z) \in \mathcal{A}$ such that $\phi(s, z) > 0$ and $w^h(s, z) > 0$. Therefore there exists an open set $\mathcal{U} \subset \mathcal{A}$, such that $w^h|_{\mathcal{U}} > 0, \phi|_{\mathcal{U}} > 0$, since w^h and ϕ are continuous. Without loss of generality we may assume that $\mathcal{A} = \mathcal{U}$. Thus

$$\langle \phi, w^h \rangle_{L^2(\mathcal{A})} > 0.$$

We define a function $f := \square_g(\kappa^{-2}\phi) \in C_0^\infty(B_{t_0})$. Then it holds that $w^f = \kappa^{-2}\phi$ and f is essentially non-radiating. Notice, that then it holds

$$\langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)} = \langle \phi, w^h \rangle_{L^2(\mathbb{R} \times N)} = \langle \phi, w^h \rangle_{L^2(\mathcal{A})} > 0.$$

Therefore (16) is not valid either.

Next we will prove (17). Let $t_0 > 0$ be such that (15) is valid. Let $s \in (0, t_0]$. Clearly $B_s \cap \mathcal{J}^+(\mathcal{C}) = \emptyset$ and thus

$$M(s, E_1(\epsilon_1)) \cap E_2(\epsilon_2) = \emptyset.$$

Therefore we deduce that $\text{dist}_g(E_1(\epsilon_1), E_2(\epsilon_2)) \geq M$ where

$$M = \sup\{t_0 > 0 : (15) \text{ is valid}\} \in \mathbb{R}_+.$$

On the other hand for any $t_0 > M$, it holds that (15) is not valid and therefore by the definition of $\mathcal{I}^+(\mathcal{C})$ it holds that $\text{dist}_g(E_1(\epsilon_1), E_2(\epsilon_1)) \leq t_0$. Thus (17) is valid.

To see that the last equation holds, we assume that $\epsilon_1 = \epsilon_2$. Notice that

$$\text{dist}_g(E_1(\epsilon_1), E_2(\epsilon_1)) = \text{dist}_g(\overline{E_1(\epsilon_1)}, \overline{E_2(\epsilon_2)}).$$

Since sets $\overline{E_1(\epsilon_1)}$ and $\overline{E_2(\epsilon_2)}$ are compact there exist points $y_{\epsilon_1} \in \overline{E_1(\epsilon_1)}$ and $x_{\epsilon_1} \in \overline{E_2(\epsilon_2)}$ such that

$$(18) \quad d_g(y_{\epsilon_1}, x_{\epsilon_1}) = \text{dist}_g(\overline{E_1(\epsilon_1)}, \overline{E_2(\epsilon_2)}).$$

Therefore for every $n \in \mathbb{N}$ we can choose points $y_n \in \overline{E_1(1/n)}$ and $x_n \in \overline{E_2(1/n)}$ such that equation (18) holds. Since metrics d_0 and d_g give the same topological structure, it holds that $x_n \rightarrow x_0$ and $y_n \rightarrow y_0$ as $n \rightarrow \infty$ with respect to metric d_g . Thus it holds that

$$d_g(x_0, y_0) = \lim_{n \rightarrow \infty} d_g(x_n, y_n) = \text{dist}_g(E_1(1/n), E_2(1/n)).$$

This completes the proof. \square

Now we are ready to present the proof of Theorem 6.

Proof of Theorem 6. The inner products

$$\langle \kappa w^f, \kappa w^h \rangle_{L^2(\mathbb{R} \times N)}, \quad f, h \in \mathbb{D},$$

determine the same inner products for all $f, h \in C_0^\infty((0, \infty) \times \mathcal{X})$ by density. We can reverse the time to determine the inner products

$$(19) \quad \langle \kappa w^f, \kappa w^h \rangle_{L^2((0, \infty) \times N)}, \quad f, h \in C_0^\infty((0, \infty) \times \mathcal{X}).$$

Let $x, y \in \mathcal{X}$. Let $t_0 \geq 0$ and $\epsilon > 0$ be small enough. Then by Lemma 8 we can find all the essentially non-radiating functions supported in $B_\epsilon(x_0, t_0)$ as we know the inner products (19). Denote $\mathcal{C} := (0, \infty) \times B_0(y, \epsilon)$. By Lemma 9 we can find if

$$\mathcal{I}^+(\mathcal{C}) \cap B_\epsilon(x_0, t_0)$$

holds for given t_0 , and using formula (17) we find

$$\text{dist}_g(B_0(y, \epsilon), B_0(y, \epsilon)).$$

Eventually

$$d_g(x, y) = \lim_{\epsilon \rightarrow 0} \text{dist}_g(B_0(y, \epsilon), B_0(y, \epsilon)).$$

Thus inner products (19) allow us to determine the distances $d_g(x, y)$, for all $x, y \in \mathcal{X}$, and these again determine $(\mathcal{X}, g_{\mathcal{X}})$ up to an isometry (see the proof of Proposition 5 below).

Let $h \in C_0^\infty((0, \infty) \times \mathcal{X})$. We want to show that $w^h|_{(0, \infty) \times \mathcal{X}}$ can be determined from the inner products (6). As $(0, \infty) \times \mathcal{X}$ can be covered with a countable number of sets of the form $(t_0 - \epsilon, t_0) \times B$ where $t_0 > 0$, $\epsilon \in (0, t_0)$ is small and $B \subset \mathcal{X}$ is a small ball, it is enough to show that $w^h|_{(t_0 - \epsilon, t_0) \times B}$ can be determined.

By Lemma 8 we can find the set \mathcal{N} of essentially non-radiating functions $f \in C_0^\infty((t_0 - \epsilon, t_0) \times B)$ given the inner products (12). Let $f \in \mathcal{N}$. Then w^f is a solution of the following initial boundary value problem

$$(20) \quad \begin{aligned} \partial_t^2 w - \Delta_g w &= f \quad \text{in } (0, \infty) \times \mathcal{X}, \\ w|_{\mathbb{R} \times \partial \mathcal{X}} &= 0, \\ w|_{t=0} = \partial_t w|_{t=0} &= 0. \end{aligned}$$

As (\mathcal{X}, g) is known, we can solve the above equation (see for instance [27], Section 2.3.). Thus for every $f \in \mathcal{N}$ we are able to find w^f .

In particular, in the inner products

$$(21) \quad \langle w^f, \kappa^2 w^h \rangle_{L^2((0, \infty) \times \mathcal{X})}, \quad f \in \mathcal{N},$$

the left factor w^f is known. It remains to observe that for any $\phi \in C_0^\infty((t_0 - \epsilon, t_0) \times B)$ we have $w^f = \phi$ where $f = \square_g \phi \in \mathcal{N}$ and therefore the functions w^f , $f \in \mathcal{N}$, are dense in $L^2((t_0 - \epsilon, t_0) \times B)$. Hence we find $\kappa^2 w^h|_{(t_0 - \epsilon, t_0) \times B}$ from inner products (21).

Let us conclude the proof by showing that function $\kappa|_{\mathcal{X}}$ can be determined. Let $x_0 \in \mathcal{X}$ and $t_0, r > 0$ be small enough. Let \mathcal{N} be the set of essentially non-radiating functions $f \in C_0^\infty((0, t_0) \times B_g(x_0, r))$. Let $f \in \mathcal{N}$. Then we can find w^f by solving equation (20). By previous theorem we can determine the function $\kappa^2 w^f|_{(0, \infty) \times \mathcal{X}}$. Therefore we know both w^f and $\kappa^2 w^f$ for every $f \in \mathcal{N}$. Let $(\varphi_i)_{i=1}^\infty \subset C_0^\infty((0, \infty) \times \mathcal{X})$ be such a collection of positive functions that $\varphi_i \rightarrow \varphi$ in L^2 -sense where φ is the characteristic function of $(0, t_0) \times B_g(x_0, r)$. Note that that we can compute $\square_g \varphi_i := f_i \in \mathcal{N}$ and $w^{f_i} = \varphi_i$. Therefore there exists $j \in \mathbb{N}$ and $s \in (0, t_0)$ that $\varphi_j(s, x_0) > 0$. Thus we can find

$$\kappa(x_0)^2 = \frac{\kappa(x_0)^2 \varphi_j(s, x_0)}{\varphi_j(s, x_0)}.$$

Thus for every $h \in C_0^\infty((0, \infty) \times \mathcal{X})$ we can compute the mapping

$$h \mapsto w^h|_{(0, \infty) \times \mathcal{X}}.$$

□

5. THE DETERMINISTIC INVERSE PROBLEM

In this section we prove Theorem 2 in two steps: we show first that local source-to-solution map $\Lambda_{\mathcal{X}}$ determines a certain family of distance functions, and then that this family determines the geometry g . We work first under the assumption that $d_g|_{\mathcal{X} \times \mathcal{X}}$ is known, and postpone the proof that $\Lambda_{\mathcal{X}}$ determines $d_g|_{\mathcal{X} \times \mathcal{X}}$ in the end of the section. Recall that in the previous section we already determined $d_g|_{\mathcal{X} \times \mathcal{X}}$, so the step from $\Lambda_{\mathcal{X}}$ to $d_g|_{\mathcal{X} \times \mathcal{X}}$ is needed only in the proof of Theorem 2.

5.1. Reconstruction of a family of distance functions from the local source-to-solution mapping $\Lambda_{\mathcal{X}}$. By the previous section we have found the following objects:

$$(22) \quad \{(\mathcal{X}, g|_{\mathcal{X}}), d_g|_{\mathcal{X} \times \mathcal{X}}, \Lambda_{\mathcal{X}}\}$$

Here $(\mathcal{X}, g|_{\mathcal{X}})$ stands for the assumption that the Riemannian structure of the open manifold \mathcal{X} is known. We show the following theorem.

Theorem 7. *Let (N, g) be a complete Riemannian manifold. Then the local source-to-solution data (22) determines the following family of distance functions*

$$(23) \quad R_{\mathcal{X}}(N) := \{d_g(x, \cdot)|_{\mathcal{X}} : x \in N\} \subset C(\mathcal{X}).$$

This is to be proved in several steps. Let $T, \epsilon > 0$. For each $r > \epsilon$ and $x \in N$ we define a set

$$S_{\epsilon}(x, r) := (T - (r - \epsilon), T) \times B(x, \epsilon)$$

We denote for any measurable $A \subset N$ the function space

$$L^2(A) := \{u \in L^2(N) : \text{supp}(U) \subset \overline{A}\}.$$

Recall that for any $T > 0$, $f \in C_0^{\infty}(\mathbb{R}_+ \times N)$ the solution $u^f(T, \cdot) \in L^2(N)$.

Lemma 10. *Let $x, y, z \in N$, $\epsilon > 0$ and $\ell_x, \ell_y, \ell_z > \epsilon$. Then the following are equivalent:*

$$(24) \quad B(x, \ell_x) \subset \overline{B(y, \ell_y) \cup B(z, \ell_z)}$$

$$(25) \quad \begin{aligned} &\text{For all } f \in C_0^{\infty}(S_{\epsilon}(x, \ell_x)) \text{ there exists } (f_j)_{j=1}^{\infty} \subset C_0^{\infty}(S_{\epsilon}(y, \ell_y) \cup S_{\epsilon}(z, \ell_z)) \\ &\text{such that } \|u^f(T) - u^{f_j}(T)\|_{L^2(N)} \xrightarrow{j \rightarrow \infty} 0. \end{aligned}$$

Denote by $\ell_{\epsilon} = \max((\ell_z - \epsilon), (\ell_y - \epsilon))$. Here we consider that for $f \in C_0^{\infty}(S_{\epsilon}(y, \ell_y) \cup S_{\epsilon}(z, \ell_z))$ function u^f is the solution of the following wave equation:

$$(26) \quad \begin{cases} \square_g u = f, & \text{in } N \times (T - \ell_{\epsilon}, T) \\ u|_{t=T-\ell_{\epsilon}} = \partial_t u|_{t=T-\ell_{\epsilon}} = 0. \end{cases}$$

Proof. Suppose that (24) is valid. Let $f \in C_0^\infty(S_\epsilon(x, \ell_x))$, then by Theorem 2 it holds that

$$\begin{aligned} \text{supp } u^f(T) &\subset M(B(x, \epsilon), \ell_x - \epsilon) = B(x, \ell_x) \subset \overline{B(y, \ell_y) \cup B(z, \ell_z)} \\ &= \overline{M(B(y, \epsilon), \ell_y - \epsilon) \cup M(B(z, \epsilon), \ell_z - \epsilon)}. \end{aligned}$$

Notice that $B(x, \ell_x) \cap \partial(B(y, \ell_y) \cup B(z, \ell_z))$ is a set of measure zero, since by [38] $\partial B(y, \ell_y) \cup \partial B(z, \ell_z)$ is a set of measure zero. Thus in L^2 -sense

$$u^f(T, x) = \begin{cases} u^f(T, x), & x \in B(y, \ell_y) \cup B(z, \ell_z), \\ 0, & x \notin B(y, \ell_y) \cup B(z, \ell_z). \end{cases}$$

We denote $u^f(T) =: u^f$. Let χ be the characteristic function of set $B(y, r_y)$. Then we can split $u^f = \chi u^f + (1 - \chi)u^f =: u_y^f + u_z^f$. Since the boundary $\partial B(y, \ell_y)$ is a set of measure zero, it holds that $u_y^f \in L^2(B(y, \epsilon))$ and $u_z^f \in L^2(B(z, \epsilon))$. By Theorem 11 there exist sequences $(f_y^j)_{j=1}^\infty \subset C_0^\infty(S_\epsilon(y, \ell_y))$ and $(f_z^j)_{j=1}^\infty \subset C_0^\infty(S_\epsilon(z, \ell_z))$ such that sequences $(u_y^{f_y^j}(T))_{j=1}^\infty$ and $(u_z^{f_z^j}(T))_{j=1}^\infty$ converge to $u_y^f(T)$ and $u_z^f(T)$, respectively, with respect to L^2 -norm. Write $f^j = f_y^j + f_z^j \in C_0^\infty(S_\epsilon(y, \ell_y) \cup S_\epsilon(z, \ell_z))$ and let u^{f^j} be the corresponding solution of equation (26). By linearity we get that $u^{f^j} = u_y^{f_y^j} + u_z^{f_z^j}$. As $j \rightarrow \infty$ we get

$$\|u^f(T) - u^{f^j}(T)\|_{L^2(N)} \leq \|u_y^f(T) - u_y^{f_y^j}(T)\|_{L^2(N)} + \|u_z^f(T) - u_z^{f_z^j}(T)\|_{L^2(N)} \rightarrow 0$$

Suppose that (24) is not valid. Then the open set

$$U := B(x, \ell_x) \setminus \overline{B(y, \ell_y) \cup B(z, \ell_z)}$$

is not empty. The characteristic function χ_U of U is in $L^2(B(x, \ell_x))$ and by Theorem 11, it holds that there exists a sequence of sources $(f_j)_{j=1}^\infty \subset C_0^\infty(S_\epsilon(x, \ell_x))$ such that $\|u^{f_j}(T) - \chi_U\|_{L^2} \rightarrow 0$ as $j \rightarrow \infty$.

By finite speed of wave propagation, it holds that for all $f \in C_0^\infty(S_\epsilon(y, \ell_y) \cup S_\epsilon(z, \ell_z))$

$$\text{supp } u^f(T) \subset M(\ell_y - \epsilon, B(y, \epsilon)) \cup M(\ell_z - \epsilon, B(z, \epsilon)) = B(y, \ell_y) \cup B(z, \ell_z).$$

Therefore there exists $j \in \mathbb{N}$ such that

$$\inf\{\|u^{f_j}(T) - u^f(T)\|_{L^2(N)} : f \in C_0^\infty(S_\epsilon(y, \ell_y) \cup S_\epsilon(z, \ell_z))\} > 0,$$

since otherwise we could approximate χ_U with solutions u^f , $f \in C_0^\infty(S_\epsilon(y, \ell_y) \cup S_\epsilon(z, \ell_z))$. \square

For any point $p \in N$ and vector $\xi \in S_p N := \{\xi \in T_p N : \|\xi\|_g = 1\}$ we will denote the cut distance at p to direction ξ by $\tau(p, \xi)$. This is defined as

$$\tau(p, \xi) = \sup\{t > 0 : d_g(p, \gamma_{p, \xi}(t)) = t\}.$$

Let $\alpha, \beta : (0, 1) \rightarrow N$ be curves such that $\alpha(1) = \beta(0)$. Then we denote by $\alpha\beta$ the concatenated curve.

Lemma 11. *Let (N, g) be a complete Riemannian manifold. Let $x, y \in N$ and let $\gamma_{y,\xi}$ be the distance minimizing geodesic from y to x . Let $s := d_g(x, y)$. Then for all $r > 0$ the following are equivalent:*

$$(27) \quad \overline{B(x, r)} \subset B(y, r + s),$$

$$(28) \quad \tau(y, \xi) < r + s.$$

Proof. Let $r > 0$ and denote $p = \gamma_{y,\xi}(r + s)$.

Suppose that (27) is valid. Then

$$p \in \overline{B(x, r)} \subset B(y, r + s).$$

Therefore $d_g(p, y) < r + s$ and (28) must hold.

Suppose that $\tau(y, \xi) < r + s$. Then it holds that $d_g(p, y) < r + s$. By the triangle inequality it suffices to prove that

$$(29) \quad \partial B(x, r) \subset B(y, r + s).$$

Let $z \in \partial B(x, r)$. Clearly $d_g(z, y) \leq r + s$. Let α be a minimizing geodesic from x to z . Suppose first that α is not the geodesic continuation of segment $\gamma_{y,\xi}([0, s])$. Since a curve $\gamma_{y,\xi}\alpha$ has a length $r + s$ and it is not smooth at x , it must hold that $d_g(z, y) < r + s$.

On the other hand, if α is the geodesic continuation of segment $\gamma_{y,\xi}((0, s))$, it must hold that $z = \gamma_{y,\xi}(r + s) = p$. This contradicts the observation $d_g(p, y) < r + s$. Thus (29) is valid. \square

Notice that (27) is equivalent with the following:

$$(30) \quad \text{There exists } \epsilon > 0 \text{ such that } B(x, r + \epsilon) \subset B(y, r + s).$$

Next we provide an algorithm to find the cut distance function τ .

Proposition 1. *For any $y \in \mathcal{X}$ and $\xi \in S_y N$ we can find $\tau(y, \xi)$ from the local source-to-solution data (22).*

Proof. Let $y \in \mathcal{X}$. Recall that we know the distance function in \mathcal{X} and the metric tensor $g|_{\mathcal{X}}$. Therefore we can choose any $\xi \in S_y N$ and consider the geodesic segment $\gamma_{y,\xi}([0, s])$ for small values $s > 0$.

Let $s > 0$ be so small that $\gamma_{y,\xi}([0, s]) \subset \mathcal{X}$. We denote $x = \gamma_{y,\xi}(s)$. Let $r > 0$. We are interested, if

$$(31) \quad \overline{B(x, r)} \subset B(y, r + s)$$

holds. Depending on if (31) holds or not, we proceed in the following way:

- (1) If equation (31) holds, then by Lemma 11, we know that $\tau(y, \xi) < r + s$ and we choose a smaller r and test equation (31) again.
 - (a) If for all $r > 0$ equation (31) holds, then we choose a smaller $s > 0$ and do step (1) again.
- (2) If equation (31) does not hold, then we know by Lemma 11 that $\tau(y, \xi) \geq r + s$. Then we will fix $s > 0$ and choose a larger r and test equation (31) again.

Therefore

$$\tau(y, \xi) = \inf\{r + s > 0 : r, s > 0, \gamma_{y, \xi}([0, s]) \subset \mathcal{X}, (31) \text{ holds}\}.$$

Let $s, r > 0$ be fixed. Next we give a method how to see whether (31) holds or not. Notice that it is equivalent to consider, if there exists $\epsilon \in (0, r)$ so small that $B(y, \epsilon) \cup B(x, \epsilon) \subset \mathcal{X}$ and

$$M(B(x, \epsilon), r) = B(x, r + \epsilon) \subset B(y, r + s) = M(B(y, \epsilon), r + s - \epsilon).$$

Let $\epsilon > 0$ be also chosen small enough and consider an open set

$$U_\epsilon = B(x, r + \epsilon) \setminus \overline{B(y, r + s)}.$$

Let $T \geq r + s$ and χ_ϵ be the characteristic function of U_ϵ . By Approximate controllability (Thm. 11) we can approximate the function χ_ϵ in L_2 -sense with a sequence $u^{f_k}(T)$ of solutions of the wave equation (26), $f_k \in C_0^\infty((T - r, T) \times B(x, \epsilon))$. On the other hand for any $f \in C_0^\infty((T - r - s + \epsilon, T) \times B(y, \epsilon))$ it holds by Finite speed of wave propagation (Thm. 9) that $\text{supp } u^f(T) \subset B(y, r + s)$. Thus we cannot approximate the function χ_ϵ with solutions

$$\{u^f : f \in C_0^\infty((T - r - s + \epsilon, T) \times (B(y, \epsilon)))\} =: \mathcal{U}_T,$$

if U_ϵ is not an empty set. Therefore we conclude that $U_\epsilon = \emptyset$, if

$$(32) \quad \inf_{u^f \in \mathcal{U}_T} \|u^f(T) - u^h(T)\|_{L^2(N)} = 0, \quad h \in C_0^\infty(T - r, T) \times B(x, \epsilon)$$

Notice that using the Blagovestchenskii identity (see (42) in the appendix below), we can compute $\|u^f(T) - u^h(T)\|_{L^2(N)}$ for all $h \in C_0^\infty((T - r, T) \times B(x, \epsilon))$ and $u^f \in \mathcal{U}_T$ from the local source-to-solution data (22).

Denote $z = y$ and $\ell_z = \ell_y = r + s$ and $\ell_x = r + \epsilon$. With these notations as used in Lemma 10, it holds that we can approximate all the functions u^h , $h \in C_0^\infty((T - r, T) \times B(x, \epsilon))$ with functions in \mathcal{U}_T if and only if we are able to approximate the function χ_ϵ . Therefore set U_ϵ is empty if and only if (32) is valid.

We conclude that (31) holds if and only if there exists $\epsilon > 0$ such that $U_\epsilon = \emptyset$ and this is equivalent to equation (32). \square

We say that a geodesic segment $\gamma_{y, \xi}([0, s])$ is injective, if $\tau(y, \xi) > s$. In the next Lemma we will show that for every $x \in N$, there exists a injective geodesic segments emanating from \mathcal{X} that hits x .

Lemma 12. *It holds that*

$$\{\gamma_{y,\xi}(t) \in N : y \in \mathcal{X}, \xi \in S_y N, t < \tau(y, \xi)\} = N.$$

Proof. Let $p \in N$ and choose any $y \in \mathcal{X}$. Let $\gamma_{y,\xi}$ be the distance minimizing geodesic from y to p . We denote by $r = d_g(y, p)$. Then it holds that $r \leq \tau(y, \xi)$. Choose $s \in (0, r)$ such that $y_1 := \gamma_{y,\xi}(s) \in \mathcal{X}$. Let $\xi_1 := \dot{\gamma}_{y,\xi}(s)$. We will show that $r - s < \tau(y_1, \xi_1)$ and this proves the claim of this lemma.

Suppose that $\tau(y_1, \xi_1) \leq r - s$. By the symmetry of cut points, it holds that $\tau(p, \eta) \leq r - s$, where $\eta := -\dot{\gamma}_{y,\xi}(r)$. Thus there exists $t \in (0, s)$ such that for a point $z := \gamma_{y,\xi}(t)$ it holds $d_g(p, z) < r - t$. Then it should also hold that

$$d_g(y, p) \leq d_g(y, z) + d_g(z, p) < t + r - t = r.$$

This is a contradiction and therefore $r - s < \tau(y_1, \xi_1)$. \square

Notice that here the assumption \mathcal{X} is open is crucial. For instance consider the cylinder

$$\{e^{i\pi t} \in \mathbb{C} : t \in [-1, 1]\} \times (-1, 1),$$

and let $\mathcal{X} = \{1\} \times (-1, 1)$ and $p = (-1, 0)$. Then it holds that every point in \mathcal{X} is a cut point of p .

Proposition 2. *Let $z, y \in \mathcal{X}$, $\eta \in T_y \mathcal{X}$, $\|\eta\| = 1$ and $\tilde{r} < \tau(y, \eta)$. Then the local source-to-solution data (22) determines $d_g(p, z)$, where $p = \gamma_{y,\eta}(\tilde{r})$.*

Proof. Let $s \in (0, \tilde{r})$ be such that $\gamma_{y,\xi}([0, s]) \subset \mathcal{X}$. We denote by $x = \gamma_{y,\xi}(s)$. Let $r := \tilde{r} - s$. The numbers s and r will be fixed through out the rest of the proof.

Let $R > 0$. Next we will give an algorithm with which one can check whether $d_g(z, p) \leq R$ or not.

Choose $\epsilon > 0$ be small enough. By Lemma 10 the inclusion

$$(33) \quad B(x, r + \epsilon) \subset \overline{B(y, r + s) \cup B(z, R)}$$

is valid if and only if the equation (25) is valid with $\ell_x = r + \epsilon$, $\ell_y = r + s$ and $\ell_z = R$. Using the Blagovestchenskii identity (42), we can compute

$$\|u^f(T) - u^h(T)\|_{L^2(N)}$$

from the local source-to-solution data (22) for any $f \in \mathcal{U} := C_0^\infty(S_\epsilon(x, r + \epsilon))$ and $h \in \mathcal{V} := C_0^\infty(S_\epsilon(y, r + s) \cup S_\epsilon(z, R))$. Thus (25) is valid for given R and ϵ if and only if for every $f \in \mathcal{U}$ the following holds

$$\inf_{h \in \mathcal{V}} \|u^f(T) - u^h(T)\|_{L^2(N)} = 0.$$

Therefore we also know, if (33) is valid or not.

Suppose that (33) is valid. Since we assumed that $r + s < \tau(y, \xi)$, it holds that $p \in \overline{B(z, R)}$. Thus we have proved that (33) implies $d_g(p, z) \leq R$.

Therefore for all $R > 0$ and for all $\epsilon > 0$ small enough, we can verify when (33) is valid. We conclude that

$$d_g(p, z) = \inf\{R > 0 : \text{Formula (33) is valid for } R \text{ and some } \epsilon > 0\}.$$

□

Let $p \in N$ and $z \in \mathcal{X}$. By Lemma 12 it holds that there exists $y \in \mathcal{X}$ and an unit vector $\xi \in S_y N$ such that $p = \gamma_{y, \xi}(\tilde{r})$, for some $\tilde{r} < \tau(y, \xi)$. By Propositions 1 and 2 we have reconstructed $R(N)$. Therefore Theorem 7 is proved.

5.2. Reconstruction of the Riemannian manifold from the distance functions. So far we have been able to find the following *distance data*

$$(34) \quad \{\mathcal{X}, g|_{\mathcal{X}}, R_{\mathcal{X}}(N)\},$$

where $R_{\mathcal{X}}(N)$ is defined by (23). In this section we will show how one can reconstruct the topological, smooth and Riemannian structures from the distance data (34). The rest of the paper is devoted to showing the following theorem:

Theorem 8. *Let (N, g) be a complete smooth Riemannian manifold with out boundary. Let $U \subset N$ be open, bounded and have a smooth boundary. Suppose that the topological and smooth structure of U are known, and $g|_U$ is also known. Then*

$$R(N) := \{d_g(\cdot, x)|_{\overline{U}} : x \in N\} \subset C(\overline{U})$$

determines, topological, smooth and Riemannian structure of N up to isometry.

Since \overline{U} is compact, $C(\overline{U})$ is a Banach space when equipped with L^∞ -norm. We will define a mapping

$$R : N \rightarrow C(\overline{U}), \quad R(x) = r_x = d_g(x, \cdot)|_{\overline{U}}.$$

Our aim is to construct such a Riemannian structure in $R(N) \subset C(\overline{U})$ that $R : N \rightarrow R(N)$ is a Riemannian isometry.

Lemma 13. *Mapping R is continuous and one-to-one.*

Proof. Let $x, y \in N$. Then by the triangle inequality

$$\|R(x) - R(y)\|_{L^\infty(\overline{U})} = \sup_{z \in \overline{U}} |r_x(z) - r_y(z)| \leq d_g(x, y).$$

Thus R is 1-Lipschitz and therefore it is continuous.

Suppose that there exist $x, y \in N$ such that $r_x = r_y$. If $x \in \overline{U}$ then $r_y(x) = 0$ and thus $x = y$. Therefore we can assume that $x, y \in N \setminus \overline{U}$. Since \overline{U} is compact there exists a closest point $z \in \overline{U}$ to x . Then $z \in \partial U$ and it is also a closest point of \overline{U} to y . Since ∂U is smooth $n - 1$ dimensional submanifold of N , the distance minimizing

unit speed geodesic γ from z to x is orthogonal to ∂U . Since both x and y are points of the exterior of U , it holds by the uniqueness of geodesics that

$$x = \gamma(r_x(z)) = \gamma(r_y(z)) = y.$$

This completes the proof. \square

Next we will recall two topological results that allow us to prove that mapping $R : N \rightarrow R(N)$ is a homeomorphism.

Definition 4. Let X be a topological space. We say that a sequence $(x_j)_{j=1}^\infty$ in X escapes to infinity, if for every compact $K \subset X$, $x_j \in K$ for at most finitely many $j \in \mathbb{N}$.

For the proofs of two following lemmas see for instance [50].

Lemma 14. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $F : X \rightarrow Y$ be continuous. Then f is proper if and only if for every sequence $(x_j)_{j=1}^\infty \subset X$ that escapes to infinity the image sequence $(f(x_j))_{j=1}^\infty \subset Y$ escapes to infinity.

Lemma 15. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \rightarrow Y$ be, one-to-one, continuous and proper. Then mapping f is closed.

Proposition 3. Mapping $R : N \rightarrow R(N)$ is a homeomorphism.

Proof. If $\text{diam}(N) < \infty$, then N is compact, and the claim follows from basic topology. Suppose that $\text{diam}(N) = \infty$. Let $(x_j)_{j=1}^\infty \subset N$ be a sequence that escapes to infinity. Let $x_0 \in \bar{U}$. We define $X_j := \overline{B(x_0, j)}$ for every $j \in \mathbb{N}$ and $Y_j = R(X_j)$. Then $\cup_{j=1}^\infty X_j = N$ and thus

$$\lim_{j \rightarrow \infty} d_g(x_0, x_j) = \infty.$$

We will denote by $R(x_0) =: r_0$ and $R(x_j) =: r_j$. Then

$$d_\infty(r_0, r_j) \geq |d_g(x_0, x_0) - d_g(x_0, x_j)| = d_g(x_0, x_j).$$

Thus $d_\infty(r_0, r_j) \rightarrow \infty$ as $j \rightarrow \infty$. Since a compact set of a metric space is always bounded, it holds that sequence $(r_j)_{j=1}^\infty$ escapes to infinity. Therefore R is a proper mapping and by Lemma 15 it holds that R is closed. This completes the proof. \square

By Proposition 3, the topological structure of N has been found. Next we will show, how one can construct such a smooth atlases on N and $R(N)$ that mapping R is a diffeomorphism.

Let $z \in U$ and $x \in N$. Recall that $r_x := d_g(x, \cdot)|_U$ is smooth at z if and only if $z \neq x$ or $z \notin \omega(x)$ (see Lemma 2.1.11 and Theorem 2.1.14 of [29]). Where $\omega(x)$ is the cut locus of x . Using also the fact that $z \in \omega(x)$ if and only if $x \in \omega(z)$ we can

find the cut locus $\omega(z)$ from data (34). We denote by $\omega^T(z) \subset T_z N$ the tangential cut locus of z . Notice that $\omega(z)$ and $\omega^T(z)$ are closed.

We define a mapping Φ_z by

$$\Phi_z(r) := -r(z)\nabla_g r|_z \in T_z(N), \quad r \in R(N) \setminus R(\omega(z)).$$

By the following lemma it holds $\Phi_z \circ R|_{N \setminus \omega(z)} = \exp_z^{-1}$.

Lemma 16. *Let $x \in N$. We denote $B := T_z N \setminus \omega^T(z)$. Then the following are equivalent:*

$$(35) \quad \eta \in B \text{ and } \exp_z(\eta) = x$$

$$(36) \quad \nabla_g d_g(x, \cdot)|_z \in T_z N \text{ exists and } \eta = -d_g(x, z)\nabla_g d_g(x, \cdot)|_z.$$

Proof. Suppose that formula (35) is valid. Since exponential mapping \exp_z is one-to-one in the set B , point z is not in the cut locus of x and therefore function $d_g(x, \cdot)$ is smooth at z . Thus $\nabla_g d_g(x, \cdot)|_z \in T_z N$ exists and $\eta = -d_g(x, z)\nabla_g d_g(x, \cdot)|_z$. Therefore (36) is also valid.

Suppose that formula (36) is valid. Then it holds that $d_g(x, \cdot)$ is smooth at z . Thus x is not in the cut locus of z and therefore $\xi := -\nabla_g d_g(x, \cdot)|_z$ is the initial velocity of the unique distance minimizing geodesic from z to x .

$$\exp_z(\eta) = \gamma_{z,\xi}(d_g(x, z)) = x \in \exp_z(B).$$

□

We define the smooth structure on $R(N)$ by using mappings $\Phi_z, z \in U$. By Lemma 16 each mapping Φ_z is a topological coordinate mapping. Let $z, w \in U$. Then the composition

$$\Phi_z \circ \Phi_w^{-1} = (\Phi_z \circ R) \circ (\Phi_w \circ R)^{-1} = \exp_z^{-1} \circ \exp_w$$

is smooth whenever defined. Moreover, R is clearly smooth when the smooth structure of $R(N)$ is defined in this way. Therefore we have proved the following proposition.

Proposition 4. *The mapping $R : N \rightarrow R(N)$ is a diffeomorphism.*

We define a metric tensor $\tilde{g} := R_* g = (R^{-1})^* g$ on $R(N)$, that is, \tilde{g} is the push forward of g . Then $(R(N), \tilde{g})$ and (N, g) are Riemannian isometric. In the next proposition, we will provide a method to construct a local representation of \tilde{g} in local coordinates $(E_{y_i})_{i=1}^n$ of $R(N)$.

Proposition 5. *Let $\tilde{g} := R_* g$. We can construct the metric tensor \tilde{g} on $R(N)$ from the distance data (34).*

Proof. Let $r_0 \in R(N)$. We denote by $x_0 := R^{-1}(r_0)$. By Lemma 12 it holds that there exists a point $z \in U$ that is not in the cut locus of x_0 . Let $U' \subset U$ be an open neighborhood of z such that $d_g(\cdot, y)$ is smooth at x_0 for any $y \in U'$.

It holds that

$$\nabla_g d_g(\cdot, y)|_{x_0} = -\dot{\gamma}_{y, x_0}(d_g(y, x_0)) \in S_{x_0}N,$$

Let d be the exterior derivative, γ_{y, x_0} be the unique unit speed distance minimizing geodesic from y to x_0 and $S_{x_0}^*N$ is the unit cosphere at x_0 . Since U' is open and \exp_{x_0} is continuous the set $\exp_{x_0}^{-1}U' \subset T_{x_0}N$ is open. Therefore the set

$$\mathcal{V} := \{\nabla_g d_g(\cdot, y)|_{x_0} \in S_{x_0}N : y \in U'\}$$

is open in $S_{x_0}N$. Since R is a diffeomorphism it holds that

$$\mathcal{W}^* := R_*\mathcal{V}^* = \{(\nabla d_g(R^{-1}(\cdot), y)|_{r_0})^b \in S_{r_0}^*R(N) : y \in U'\}$$

is open. For any point $y \in U'$ we define an evaluation function $E_y : R(N) \rightarrow \mathbb{R}$ with the formula $E_y(r) = r(y)$. Notice that

$$dE_y|_{r_0} = (\nabla d_g(R^{-1}(\cdot), y)|_{r_0})^b,$$

and therefore

$$\mathcal{W}^* = \{dE_y|_{r_0} \in S_{r_0}^*R(N) : y \in U'\}.$$

As we know the smooth structure of $R(N)$ we can find the set \mathcal{W}^* . The last step is to show that set \mathcal{W}^* determines $\tilde{g}(r_0)$.

Let

$$\mathbb{R}_+\mathcal{W}^* := \{sv \in T_{r_0}^*R(N) : v \in \mathcal{W}^*, s > 0\}$$

be the open cone generated by \mathcal{W}^* . Let $\{E_j\}_{j=1}^n$ be a local coordinate system at r_0 . For any $s > 0$ and $v \in \mathcal{W}^*$ it holds in coordinates $\{E_j\}_{j=1}^n$ that

$$F(sv) := \tilde{g}^{ij}(r_0)v_iv_j = s^2.$$

Thus we know the function $F : \mathbb{R}_+\mathcal{W}^* \rightarrow \mathbb{R}$. Since $\mathbb{R}_+\mathcal{W}^*$ is open, we get

$$\tilde{g}^{ij}(r_0) = \frac{\partial}{\partial E_i} \frac{\partial}{\partial E_j} F.$$

□

By Propositions 3, 4 and 5 it holds that we can reconstruct $(R(N), \tilde{g})$, and that (N, g) and $(R(N), \tilde{g})$ are isometric as Riemannian manifolds. Thus we have proved Theorem 8.

Now we are ready to prove Theorem 1. Given data (2) we use Theorems 3-8 to reconstruct $(R(N), \tilde{g})$. Notice that, due to the reversion of time and time translations, knowing the inner products for the solutions of the time reversed wave equation (5), is equivalent for knowing the inner products for the solutions of the wave equation (11). As discussed in the beginning of Section 4 we may replace the \mathcal{X} with smaller

set with smooth boundary as needed for the proof of Theorem 6. Also in the proof of Theorem 8 we choose a small closed Riemannian ball $\overline{U} \subset \mathcal{X}$ such that ∂U is smooth.

In order to prove Theorem 2 we still need some preparations.

Lemma 17. *Let (N, g) and \mathcal{X} be as in the formulation of Theorem 2. Then data $\{\mathcal{X}, \Lambda_{\mathcal{X}}\}$ determines the distance function $d_g : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ I.e. $d_g|_{\mathcal{X} \times \mathcal{X}}$.*

Proof. Let $x, y \in \mathcal{X}$. Since \mathcal{X} is a smooth manifold, we may choose an auxiliary metric d_0 on \mathcal{X} that gives the same topology as g . Let $\epsilon > 0$ and consider the metric ball $B_{d_0}(x, \epsilon)$. We will denote by $\mathcal{B}_{\epsilon} := (0, \infty) \times B_{d_0}(x, \epsilon)$ and

$$\mathcal{Y}_{\epsilon} = \{f \in C_0^{\infty}(\mathcal{B}_{\epsilon}) : \text{there exists } t \geq 0 \text{ s.t. } \text{supp}(\Lambda_{\mathcal{X}}f)(t, \cdot) \cap B_{d_0}(y, \epsilon) \neq \emptyset\}.$$

Notice that $\mathcal{Y}_{\epsilon} \neq \emptyset$. Let $f \in \mathcal{Y}$. We define

$$t_{\epsilon}^f = \inf\{t > 0 : \text{supp}(\Lambda_{\mathcal{X}}f)(t, \cdot) \cap B_{d_0}(y, \epsilon) \neq \emptyset\}.$$

By the finite speed of wave propagation (Thm 9) the following holds

$$t_{\epsilon}^f \geq \text{dist}_g(B_{d_0}(x, \epsilon), B_{d_0}(x, \epsilon)).$$

Define $t_{\epsilon} := \inf_{f \in \mathcal{Y}} t_{\epsilon}^f$. By the approximate controllability (Thm 11) the equality

$$t_{\epsilon} = \text{dist}_g(B_{d_0}(x, \epsilon), B_{d_0}(x, \epsilon))$$

holds. Thus the following limit is valid

$$d_g(x, y) = \lim_{\epsilon \rightarrow 0} t_{\epsilon}.$$

This completes the proof. \square

In the next lemma we will show, how one can construct a smooth local coordinate system with distance functions.

Lemma 18. *Let (N, g) be a smooth complete Riemannian manifold and $p \in N$. Suppose that there exist an open neighborhood U of p and an open set $V \subset N$ such that $d|_{U \times V}$ is smooth. Then there exists points y_1, \dots, y_n , where $n = \dim N$ such that $\{d_g(\cdot, y_j)\}_{j=1}^n$ is a smooth coordinate system around at p .*

Proof. Choose any point $q \in V$. Since $d|_{U \times V}$ is smooth, it holds that q is not a cut point of p . Let $v \in T_p N$ be such that $\exp_p(v) = q$. Then $D(\exp_p)|_v$ is invertible. Let $(\eta_i)_{i=1}^n$ be any orthonormal basis of $T_q N$. Then it holds that vectors $v_i := D(\exp_p^{-1})\eta_i$ are a basis of $T_p N$. Consider curves

$$s \mapsto c_i(s) := \exp_p^{-1}(\gamma_{q, \eta_i}(s)) \in T_p N.$$

Since curves c_i satisfy initial conditions

$$c_i(0) = v \text{ and } \dot{c}_i(0) = v_i,$$

it holds that for a small $s_0 > 0$ the vectors $(c_i(s_0))_{i=1}^\infty \subset T_p N$ are linearly independent. We denote $y_i := \gamma_{q, \eta_i}(s_0)$. Then it holds that gradients

$$\nabla_g d_g(y_i, \cdot)|_p = -\frac{c_i(s_0)}{\|c_i(s_0)\|_g}$$

are linearly independent. By the Inverse function theorem we have that there exists a neighborhood $U' \subset U$ of p such that mapping

$$x \mapsto (d_g(y_j, x))_{j=1}^n \in \mathbb{R}^n, \quad x \in U'$$

is a smooth coordinate mapping. \square

For any $z \in \overline{U}$ there exists an *evaluation function* $E_z : R(N) \rightarrow \mathbb{R}$ defined by formula $E_z(r) = r(z)$.

Lemma 19. *For every $r_0 \in R(N)$ there exists points $(y_i)_{i=1}^n \subset U$ such that function*

$$r \mapsto (E_{y_i}(r)), \quad r \in R(N)$$

is a local smooth coordinate system near r_0 .

Proof. Let $x_0 = R^{-1}(r_0)$. Choose $U' \subset U$ such that $r_0|_{U'}$ is smooth. By Lemma 18 there exists points $(y_i)_{i=1}^n \subset U'$ such $q \mapsto (d_g(y_i, q))_{i=1}^n$ is a smooth local coordinate mapping at x_0 . Since

$$(E_{y_i}(r))_{i=1}^n = (r(y_i))_{i=1}^n = (r_{y_i}(R^{-1}(r)))_{i=1}^n = (d_g(y_i, R^{-1}r))_{i=1}^n,$$

it holds that $r \mapsto (E_{y_i}(r))_{i=1}^n$ is a smooth local coordinate mapping at r_0 . Therefore we have proved the claim. \square

Now we are finally ready to give a proof for Theorem 2.

Proof of Theorem 2. By making \mathcal{X}_i smaller, if needed, we may assume without lost of generality that \mathcal{X}_i is precompact with smooth boundary and that $\phi : \overline{\mathcal{X}}_1 \rightarrow \overline{\mathcal{X}}_2$ (see (4)) is a diffeomorphism. Denote $R(N_i) = \{d_{g_i}(x, \cdot)|_{\overline{\mathcal{X}}_i} : x \in N_i\}$ and consider a mapping

$$R_i : N_1 \rightarrow R(N_i), \quad i = 1, 2.$$

By Lemma 17 it holds that the assumptions of Theorem 7 are valid. Therefore it holds that Theorem 8 is also valid. Thus we can find mapping R_i and in particular mappings

$$\Phi : C(\overline{\mathcal{X}}_1) \rightarrow C(\overline{\mathcal{X}}_2), \quad \Phi(f) = f \circ \phi^{-1} \quad \text{and} \quad \Psi := R_2^{-1} \circ \Phi \circ R_1 : N_1 \rightarrow N_2$$

are well defined.

We start with the following observation. Let $x \in \mathcal{X}_1$ and denote $y := \phi(x)$. Then

$$\Psi(x) = (R_2^{-1} \circ \Phi \circ R_1)(x) = y.$$

Since

$$(\Phi \circ R_1)(x)(y) = \Phi(R_1(x))(y) = (R_1(x) \circ \phi^{-1})(y) = R_1(x)(x) = d_{g_1}(x, x) = 0$$

and for any $z \in \mathcal{X}_2$

$$0 = d_{g_2}(z, y) = R_2(z)(y)$$

holds if and only if $z = y$. Therefore $\Psi|_{\mathcal{X}_1} = \phi$

Next we note that mapping Φ is a homeomorphism with inverse $\Phi^{-1}(f) = f \circ \phi$, $f \in C(\overline{\mathcal{X}_2})$. Therefore Ψ is a homeomorphism. Since Ψ is one-to-one onto it holds that

$$(37) \quad R_2(N_2) = (\Phi \circ R_1)(N_1).$$

By the Lemma 19 and Section 2.3 (with minor modifications) of [34] we can find such a local coordinates for $R_1(N_1)$ and $R_2(N_2)$ in which mapping Φ is an identity mapping of \mathbb{R}^n . Therefore $\Phi : R_1(N_1) \rightarrow R_2(N_2)$ is smooth and thus Ψ is diffeomorphism.

The pullback $\tilde{g}_2 := \Psi^* g_2$ is a Riemannian metric tensor on N_1 . Since $\tilde{g}_2|_{\overline{\mathcal{X}_1}} = \phi^*(g_2|_{\overline{\mathcal{X}_2}})$, it holds by the proof of Lemma 17 and Proposition 5 that $\tilde{g}_2|_{\overline{\mathcal{X}_1}} = g_1|_{\overline{\mathcal{X}_1}}$. Then the claim $\tilde{g}_2 = g_1$ follows from the Section 2.4 of [34] with minor modifications. \square

6. APPENDIX

6.1. Propagation of waves on a Riemannian manifold. Let (N, g) be a smooth complete Riemannian manifold with out boundary. Let $T > 0$ and $B \subset N$ be open and bounded. Write

$$\mathcal{F}_{B,T} := \{f \in C_0^\infty(\mathbb{R} \times N) : \text{supp } f \subset (0, T) \times B\}.$$

In this section we will consider the behavior the solution of following wave equation:

$$(38) \quad \begin{cases} (\partial_t^2 - \Delta_g)u = f, & \text{in } (0, \infty) \times N \\ u|_{t=0} = \partial_t u|_{t=0} = 0, & f \in \mathcal{F}_{B,T}. \end{cases}$$

We will use the following notation \tilde{N} for the product manifold $\tilde{N} := \mathbb{R} \times N$. Let $p \in N$ and $a > 1$. We define a set $C_{p,T}$ to be the cone

$$C_{p,T} := \{(t, q) \in \tilde{N} : 0 \leq t \leq T, d_N(p, q) \leq T - t\}$$

and $C_{p,T,a}$ to be the cone

$$C_{p,T,a} := \{(t, q) \in \tilde{N} : 0 \leq t \leq \frac{T}{a}, d_N(p, q) \leq T - at\}.$$

Theorem 9 (Finite speed of propagation). *Let $f \in L^2(\tilde{N})$. Suppose that u solves*

$$\begin{cases} (\partial_t^2 - \Delta_g)u = f, & \text{in } (0, \infty) \times N \\ f|_{C_{p,T}} = 0 \\ u|_{B(p,T) \times \{t=0\}} = \partial_t u|_{B(p,T) \times \{t=0\}} = 0, \end{cases}$$

Then

$$u|_{C_{p,T}} = 0.$$

Proof. See [45]. □

Next we will provide a useful corollary for Theorem 9.

Corollary 2. *Let $B \subset N$ be open and bounded. Suppose that u solves*

$$\begin{cases} (\partial_t^2 - \Delta_g)u = f, & \text{in } (0, \infty) \times N \\ f \in C_0^\infty((0, \infty) \times B) \\ \text{supp } u(0, \cdot) \subset B \\ \text{supp } \partial_t u(0, \cdot) \subset B \end{cases}$$

Then for all $T \in (0, \infty)$ the following holds

$$\text{supp } u(T) \subset M(T, B) \text{ and } \text{supp } \partial_t u(T) \subset M(T, B)$$

Proof. Let $T > 0$. The set

$$\text{supp } u(0, \cdot) \bigcup \text{supp } \partial_t u(0, \cdot) =: K_0 \subset B.$$

is bounded and closed, therefore it is compact. Since $f \in C_0^\infty((0, \infty) \times B)$, we can find a compact set $K'_T \subset B$ such that for every $0 \leq t \leq T$ it holds

$$\text{supp } f(\cdot, t) \subset K'_T.$$

Let

$$K_T := K_0 \cup K'_T \subset B.$$

Take any $p \in N \setminus K_T$ such that $d_g(p, \partial K_T) \geq T$. By assumption it holds

$$u|_{B_N(p,T) \times \{t=0\}} = \partial_t u|_{B_N(p,T) \times \{t=0\}} = 0 \text{ and } f|_{C_{p,T}} \equiv 0.$$

By Theorem 9 we have

$$u|_{C_{p,T}} \equiv 0.$$

Especially we have $u(p, t) = 0$ for all $t \in [0, T]$. Therefore it holds that $u(p, t) = 0$ for all $p \in N \setminus K_T$ and $t \in [0, T]$ such that $d_g(p, \partial K_T) \geq T$. Thus we have proved that for all $t \in [0, T]$

$$u(t, \cdot)|_{N \setminus M(T, K_T)} \equiv 0$$

and therefore it holds that

$$\text{supp } u(T, \cdot) \subset \overline{M(T, K_T)} \subset M(T, B)$$

and

$$\text{supp } \partial_t u(T, \cdot) \subset \overline{M(T, K_T)} \subset M(T, B).$$

□

Consider an open double cone created by a cylindrical set $(0, 2T) \times B$

$$C(T, B) = \left\{ (x, t) \in (0, 2T) \times N : \begin{array}{l} d_g(x, B) < t, t \leq T \\ d_g(x, B) < 2T - t, t \geq T \end{array} \right\}$$

Theorem 10 (Tataru's Unique continuation theorem). *Let $B \subset N$ be open and bounded. Let $u \in C_0^\infty(\mathbb{R} \times N)$. Suppose that $(\partial_t^2 - \Delta_g)u = 0$ in $(0, 2T) \times M(T, B)$ and $u|_{(0, 2T) \times B} \equiv 0$. Then $u|_{C(T, B)} \equiv 0$.*

Proof. See for instance [44, 27].

□

We denote by

$$L^2(A) := \{f \in L^2(N) : V_g\{x \in (N \setminus A) : f(x) \neq 0\} = 0\}$$

for any measurable set $A \subset N$.

Theorem 11 (Approximate controllability). *Let $B \subset N$ be open and bounded. For any $T > 0$ set*

$$\mathcal{U}_T := \{u^f(T) : f \in \mathcal{F}_{B, T}\}$$

is dense in Hilbert space $L^2(M(T, B))$.

Proof. By Finite speed of wave propagation $\mathcal{U}_T \subset L^2(M(T, B))$. Since $L^2(M(T, B))$ is a Hilbert space, it suffices to prove that $\mathcal{U}_T^\perp = \{0\}$. Suppose that $\phi \in L^2(M(T, B))$ is such that $(u^f(T), \phi)_{L^2(N)} = 0$ for all $f \in \mathcal{F}_{B, T}$. Let $w \in C^\infty(\mathbb{R} \times N)$ be the unique solution of the following wave equation:

$$(39) \quad \begin{cases} (\partial_t^2 - \Delta_g)w = 0, & \text{in } (0, T) \times N \\ w|_{t=T} = 0, \quad \partial_t w|_{t=T} = \phi. \end{cases}$$

Let $f \in \mathcal{F}_{B, T}$. By Corollary 2, it holds that there exists a compact set of N that contains the $\text{supp } u^f(t)$ for each $t \in (0, T)$. We use the Green identities for the following

$$\begin{aligned} (f, w)_{L^2((0, T) \times N)} &= (\Box u^f, w)_{L^2((0, T) \times N)} - (u^f, \Box w)_{L^2((0, T) \times N)} \\ &= \int_0^T \int_N \Box u^f w - u^f \Box w \, dV_g dt \\ &= \int_0^T \int_N \partial_t^2 u^f w - u^f \partial_t^2 w \, dV_g dt - \int_0^T \int_N \Delta_g u^f w - u^f \Delta_g w \, dV_g dt \end{aligned}$$

$$\begin{aligned}
&= \int_N \int_0^T \partial_t^2 u^f w - u^f \partial_t^2 w \, dt dV_g = \int_N [u^f \partial_t w - w \partial_t u]_0^T dV_g \\
&= \int_N u^f(T) \partial_t w(T) - w(T) \partial_t u(T) - (u^f(0) \partial_t w(0) - w(0) \partial_t u(0)) dV_g \\
&\stackrel{(38)(39)}{=} \int_N u^f(T) \phi \, dV_g = 0
\end{aligned}$$

Therefore we got that $(f, w)_{L^2((0,T) \times N)} = 0$ for all $f \in \mathcal{F}_{B,T}$. Since $\mathcal{F}_{B,T}$ is dense in $L^2((0,T) \times B)$, it holds that $w \equiv 0$ in $(0, T] \times B$.

Next we consider the following wave equation:

$$(40) \quad \begin{cases} (\partial_t^2 - \Delta_g)W = 0, & \text{in } (0, 2T) \times N \\ W|_{t=0} = w(0), \quad \partial_t W|_{t=0} = \partial_t w|_{t=0}. \end{cases}$$

Let W be the unique solution of (40). By the uniqueness of solutions, it must hold that $W|_{[0,T] \times N} = w$. On the other hand function $\tilde{w}(t, x) = -w(2T - t, x)$ solves the wave equation

$$(41) \quad \begin{cases} (\partial_t^2 - \Delta_g)\tilde{w} = 0, & \text{in } (T, 2T) \times N \\ \tilde{w}|_{t=T} = 0, \quad \partial_t \tilde{w}|_{t=T} = \phi, \end{cases}$$

since

$$\tilde{w}(T, x) = -w(2T - T, x) = 0 \text{ and } \partial_t \tilde{w}|_{t=T} = \partial_t w(2T - T) = \phi.$$

Thus $\tilde{w} \equiv 0$ in $[T, 2T) \times B$. By the choice of initial conditions of (40), it holds that W also solves (41) and therefore we must have again by the uniqueness of solutions that $W|_{[T, 2T) \times B} = \tilde{w} \equiv 0$. Thus we have proved the following: Function W solves $\square W = 0$ in $(0, 2T) \times N$ and $W|_{(0, 2T) \times B} \equiv 0$.

By Unique continuation (Theorem 10), it holds that $W|_{C(T, B)} \equiv 0$. Since $M(T, B) \times \{T\} \subset C(T, B)$ we have

$$\phi|_{M(T, B)} = \partial_t W|_{t=T}|_{M(T, B)} = 0.$$

□

6.2. Blagovestchenskii identity. In this section our aim is to prove the Blagovestchenskii identity on a complete Riemannian manifold (N, g) . That is the claim of the following theorem.

Theorem 12. *Let (N, g) be a complete Riemannian manifold. Let $T > 0$, $B \subset N$ be open and bounded. Let*

$$\Lambda_{B, 2T} : \mathcal{F}_{B, 2T} \rightarrow C^\infty((0, 2T) \times B), \quad \Lambda_{B, 2T}(f) = u^f|_{(0, 2T) \times B},$$

be the local source-to-solution mapping of the following initial value problem

$$\begin{cases} (\partial_t^2 - \Delta_g)u = f, & \text{in } (0, \infty) \times N \\ u|_{t=0} = \partial_t u|_{t=0} = 0, & f \in \mathcal{F}_{B, 2T}. \end{cases}$$

Let $f, h \in \mathcal{F}_{B,2T}$, then the inner product $(u^f(T), u^h(T))_{L^2(N)}$ is given by the Blagovestchen-skii identity

$$(42) \quad (u^f(T), u^h(T))_{L^2(N)} = \int_0^T (f(t), (J\Lambda_{B,2T} - \Lambda_{B,T}^* J)h(t))_{L^2(B)} dt,$$

where operator $J : L^2(0, 2T) \rightarrow L^2(0, T)$ is defined as

$$J\phi(t) = \frac{1}{2} \int_t^{2T-t} \phi(s) ds.$$

The proof of this theorem is postponed for later. Let $B \subset N$ be open and bounded. Suppose that we know the solution mapping Λ of problem (38) restricted to $B \times (0, 2T)$ that is

$$\Lambda_{B,2T} : C_0^\infty(B \times (0, 2T)) \rightarrow C^\infty(B \times (0, 2T)), \quad \Lambda_{B,2T} f = u^f|_{B \times (0, 2T)},$$

where u^f is the unique solution of (38).

Let $f, h \in \mathcal{F}_{B,2T}$ and consider a mapping $w : [0, 2T] \times [0, 2T] \rightarrow \mathbb{R}$,

$$w(t, s) = (u^f(t), u^h(s))_{L^2(N)}.$$

We emphasise that we cannot apriori compute $w(t, s)$, since for any $t \in (0, 2T)$ we only know $u^f(t)|_B$. We will use a one dimensional wave equation to compute $w(t, s)$. We consider s as a spatial variable and t as a time variable. We apply the definition of $\Lambda_{B,2T}$ and the knowledge that u^f, u^h are solutions of (38). Then we get

$$\begin{aligned} (\partial_t^2 - \partial_s^2)w(t, s) &= (\partial_t^2 - \partial_s^2)(u^f(t), u^h(s))_{L^2(N)} \\ &= (\partial_t^2 u^f(t), u^h(s))_{L^2(N)} - (u^f(t), \partial_s^2 u^h(s))_{L^2(N)} \\ &\stackrel{(38)}{=} (\Delta_g u^f(t) + f(t), u^h(s))_{L^2(N)} - (u^f(t), \Delta_g u^h(s) + h(s))_{L^2(N)} \\ &= (\Delta_g u^f(t), u^h(s))_{L^2(N)} - (u^f(t), \Delta_g u^h(s))_{L^2(N)} \\ &\quad + (f(t), u^h(s))_{L^2(N)} - (u^f(t), h(s))_{L^2(N)} \\ &\stackrel{*}{=} (f(t), u^h(s))_{L^2(N)} - (u^f(t), h(s))_{L^2(N)} \\ &= (f(t), (\Lambda_{B,2T} h)(s))_{L^2(B)} - ((\Lambda_{B,2T} f)(t), h(s))_{L^2(B)} := F(t, s) \end{aligned}$$

Here at $*$ we used the Greens formula. Notice that there is no boundary terms due the Finite speed of wave propagation. The important thing is that the function $(t, s) \mapsto F(t, s)$ can be computed, if the solution mapping $\Lambda_{B,2T}$ is given. By (38) it holds that

$$w(0, s) = 0 = \partial_t w(t, s)|_{t=0}.$$

Thus w is a solution of the following $(1+1)$ -dimensional initial value problem:

$$(43) \quad \begin{cases} (\partial_t^2 - \partial_s^2)w = F, & \text{in } (0, 2T) \times \mathbb{R} \\ w|_{t=0} = \partial_t w|_{t=0} = 0. \end{cases}$$

Recall that the following formula

$$(44) \quad w(t, s) = \frac{1}{2} \int_0^t \int_{s-\tau}^{s+\tau} F(t - \tau, y) dy d\tau, \quad s \in \mathbb{R}, \quad t \in [0, 2T],$$

solves (43) (See [15]). We use equation (44) to compute

$$w(T, T) = \frac{1}{2} \int_0^T \int_{T-s}^{T+s} F(T - s, y) dy ds.$$

Apply the change of variables $T - s = \tau$

$$w(T, T) = \frac{1}{2} \int_T^0 \int_{\tau}^{2T-\tau} -F(\tau, y) dy d\tau = \frac{1}{2} \int_0^T \int_{\tau}^{2T-\tau} F(\tau, y) dy d\tau.$$

Next we pluck in the definition of $F(t, s)$ to previous formula. Then

$$\begin{aligned} w(T, T) &= \frac{1}{2} \int_0^T \int_t^{2T-t} (f(t), \Lambda_{B,2T}(h)(s))_{L^2(B)} - (\Lambda_{B,2T}(f)(t), h(s))_{L^2(B)} ds dt. \\ &= \int_0^T (f(t), J\Lambda_{B,2T}h(t))_{L^2(B)} - (\Lambda_{B,2T}(f)(t), Jh(t))_{L^2(B)} dt. \\ &= (f, J\Lambda_{B,2T}h)_{L^2(B \times (0,T))} - (\Lambda_{B,2T}f, Jh)_{L^2(B \times (0,T))}. \end{aligned}$$

Here $J : L^2(0, 2T) \rightarrow L^2(0, T)$

$$J\phi(t) = \frac{1}{2} \int_t^{2T-t} \phi(s) ds.$$

We define the mapping $R : L^2(0, T) \rightarrow L^2(0, T)$ as the reversion of time that is

$$R\phi(t) = \phi(T - t).$$

Lemma 20. *The adjoint mapping of $\Lambda_{B,2T}$ in $L^2((0, T) \times B)$ is $\Lambda_{B,2T}^* = \Lambda_{B,T}^* = R\Lambda_{B,T}R$.*

Proof. Let $f, h \in \mathcal{F}_{B,T}$ and consider the wave equations

$$\begin{cases} (\partial_t^2 - \Delta_g)w = f, & \text{in } (0, T) \times N \\ w|_{t=0} = \partial_t w|_{t=0} = 0 \end{cases} \quad \text{and} \quad \begin{cases} (\partial_t^2 - \Delta_g)u = h, & \text{in } (0, T) \times N \\ u|_{t=T} = \partial_t u|_{t=T} = 0. \end{cases}$$

Since the solution w depends only on the initial values, it is clear that $\Lambda_{B,2T}^* = \Lambda_{B,T}^*$. We start with observing that

$$(f, u)_{L^2((0,T) \times N)} - (w, h)_{L^2((0,T) \times N)} = 0.$$

This holds due the computations we have done in the proof of Theorem 11. Notice that we can write

$$(f, u)_{L^2((0,T) \times B)} - (\Lambda_{B,T}f, h)_{L^2((0,T) \times B)} = (f, u)_{L^2((0,T) \times B)} - (w, h)_{L^2((0,T) \times B)} = 0$$

Then it follows:

$$\Lambda_{B,T}^* h = u|_{(0,T) \times B}.$$

Now replace $f = Rh$. Then

$$\square Ru = \square u(T - \cdot, \cdot) = h(T - \cdot, \cdot) = Rh \text{ and } Ru(0, \cdot) = \partial_t Ru(0, \cdot) = 0.$$

By the uniqueness of solution it holds that

$$Ru|_{(0,T) \times B} = w|_{(0,T) \times B} = \Lambda_{B,T} f = \Lambda_{B,T} Rh.$$

Since $R \circ R = id_{L^2((0,T) \times B)}$ we get

$$u|_{(0,T) \times B} = R\Lambda_{B,T}Rh.$$

Thus we have proved that

$$R\Lambda_{B,T}R = \Lambda_{B,T}^*.$$

□

Now we are ready to present the proof of Theorem 12.

Proof of Theorem 12. Let $f, h \in \mathcal{F}_{B,2T}$. By the computations done earlier in this section it holds that

$$(u^f(T), u^h(T))_{L^2(N)} = (f, J\Lambda_{B,2T}h)_{L^2(B \times (0,T))} - (\Lambda_{B,2T}f, Jh)_{L^2(B \times (0,T))}.$$

By Lemma 20 we have $\Lambda_{B,2T}^* = R\Lambda_{B,T}R$. Therefore we conclude that equation (42) is valid. □

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